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Stationarity of the strain energy density for some classes of anisotropic solids

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Abstract

Homogeneous, anisotropic and linearly elastic solids, subjected to a given state of strain (or stress), are considered. The problem dealt with consists in finding the mutual orientations of the principal directions of strain to the material symmetry axes in order to make the strain energy density stationary. Such relative orientations are described through three Euler's angles. When the stationarity problem is formulated for the generally anisotropic solid, it is shown that the necessary condition for stationarity demands for coaxiality of the stress and the strain tensors. From this feature, a procedure which leads to closed form solutions is proposed. To this end, tetragonal and cubic symmetry classes, together with transverse isotropy, are carefully dealt with, and for each case *all* the sets of Euler's angles corresponding to critical points of the energy density are found and discussed. For these symmetries, three material parameters are then defined, which play a crucial role in ordering the energy values corresponding to each solution.

Keywords: Anisotropic elasticity; Strain energy density; Tetragonal symmetry; Transverse isotropy; Cubic symmetry

1. Introduction

Minimization of the strain energy density is of considerable significance when stiff structures or structured materials must be achieved for a given loading, whereas its maximization is an outstanding feature when a large amount of energy absorption under impact loading is demanded. Contrary to isotropic solids, in presence of elastic anisotropy the strain energy density changes when any material element is rotated to the principal directions of stress or strain. Accordingly, the orientation of the material axes can be employed as design variable to achieve the desired maximum or minimum value of the strain energy density. In designing living tissues, nature somehow employs this kind of strategy, and adjusts the microstructure of the material (i.e., its anisotropy), to enhance the mechanical performances. On the other hand, the same

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idea is artificially adopted when some man-made materials are produced. Among these, fibrous composites represent the most common example of materials intrinsically anisotropic and susceptible to be properly designed for given purposes.

The aim of this paper is to rationalize the problem of finding the extrema for the strain energy density, with reference to linear elastic solids in presence of material symmetries.

Referring to a linearly elastic anisotropic solid, defined by an elasticity tensor with components C_{ijhk}, subjected to a constant strain state characterized by given principal strains, this goal can be achieved by answering to the following questions: (a) which conditions must be satisfied by the stress and the strain fields to make the strain energy density stationary, and (b) which are explicitly the corresponding mutual orientations of the strain and the elasticity tensors that satisfy these conditions?

The answer to the first question is partially known. The results obtained up to now, which will be briefly reviewed later, concern essentially the determination of qualitative conditions to be satisfied by absolute maxima and minima for the strain energy density, and the number of such critical points. The problem of the explicit evaluation of the orientations corresponding to all the stationarity values of the strain energy density has only partially been solved. On the last point is focused the main task of the present paper, where for some classes of anisotropy (namely, tetragonal system, transverse isotropy and cubic symmetry) *all* the orientations of the principal directions of strain to the material symmetry axes at the critical points are found and discussed.

The main results on the subject appeared in the literature are reviewed in Section 2, whereas Section 3 deals with the general formulation of the problem. Use is made of a proper definition of the elasticity tensor in six dimensions which, contrary to the classical Voigt's representation, preserves the tensorial character of the constitutive law. The mutual orientation of the principal directions of strain to the material symmetry axes is then described through three Euler's angles. This choice, despite a certain inherent formal complexity of the equations governing the problem, turns out to be appropriate when the orientations corresponding to stationary values of the energy density are sought explicitly. The general condition for stationarity of the strain energy density is also revisited, and it is shown that critical points are characterized by *coaxiality of the stress and strain tensors*. Such a feature is then at the origin of the solution procedure proposed here, which consists in finding the Euler's angles that render a certain system of linear equations singular.

Explicit values of these angles are then found in Section 4, with reference to solids with tetragonal symmetry. These results are then specialized in Section 5 to the case of transverse isotropy, and in Section 6 to the case of cubic symmetry.

The results obtained are then summarized in Section 7, where some concluding remarks are also made.

2. An account of the literature

Pioneering works where extreme values of the strain energy density in anisotropic bodies are sought are those by Banichuk (1981, 1983). Here, the problem of simultaneously evaluating the most efficient shapes for anisotropic rods in torsion and the orientation of the anisotropy axes which minimize the structural compliance is dealt with. The problem of defining the local values of the elastic coefficients, with fixed directions of material axes, which minimize the energy density is also considered in plane elasticity. These results have been extended in Banichuk and Kobelev (1987) to the case of ideally elastic–plastic solids. Anisotropic plates with variable elastic moduli and material axes orientation have been also studied by Kartvelishvili and Kobelev (1984), referring to optimal design for compliance and natural vibrational frequency.

Beside these structural formulations, the study of the best positioning of elastic symmetry planes in three-dimensional orthotropic bodies for minimum potential energy of deformation has been carried out in a general way in Seregin and Troitskii (1981). In this work, through the application of the Lagrangian multipliers method, it is shown that the solution is locally characterized by a mechanically meaningful condition, that is, *coaxiality of the stress and strain tensors*. Contrary to isotropic elasticity, where the strain and stress tensors are always coaxial, in anisotropic elasticity this feature is, in general, lost. The non-trivial result obtained by Seregin and Troitskii emphasizes a requirement that must always be fulfilled when extreme values of the global stiffness are sought; consequently, it should be assumed as a guidance for an optimal spatial arrangement of the material symmetry axes.

Later, but independently, the same problem has been dealt with in Rovati and Taliercio (1991, 1993) where orientations of the material symmetry axes which maximize or minimize the global elastic stiffness of a generally anisotropic three-dimensional continuum are sought. Necessary stationarity conditions for the strain energy density are directly computed, assuming the strain state to be given, and their mechanical interpretation (that is, collinearity of principal directions of stress and strain) is highlighted. Some closed form solutions for cubic and transversely isotropic materials are found, and a material parameter, responsible of the relative shear stiffness of the solid, is introduced. It is shown how two classes of solutions can be defined according to its value: one, where stationarity of the strain energy density is accompanied by full collinearity of principal directions of stress, strain and material axes; the other one, where this collinearity is only partially preserved.

Due to pertinence to practical applications, much effort has been devoted to two-dimensional solids. In particular, the elastic problem previously described has been reformulated for plane elasticity in Sacchi Landriani and Rovati (1991), and conditions for absolute maximum and minimum structural stiffness are found; an extension to plates in bending is given as well. Careful investigations in this direction should be mentioned, such as those given by Pedersen (1989), where it is found that the best orientations of the material axes depend on a dimensionless material parameter, plus the ratio of the two principal strains. Coaxiality of the material axes and the principal strain directions always corresponds to stationary values for the energy density (trivial solutions); however, in some strain conditions, stationarity can also be achieved at some non-trivial orientations. In addition to these considerations referred to any material point, analyses are also carried out for the whole solid (Pedersen, 1990), through applications of sensitivity analysis, finite element analysis, and optimization procedures. Homogenization techniques, coupled with finite element analyses and design for optimal structural performances, have led to the very effective method of topology optimization (see Eschenauer and Olhoff, 2001, and the references therein).

A modern formulation of the problem of finding the best orientations of the material symmetry axes in a three-dimensional continuum is given by Banichuk (1996), where the application of spectral methods of tensor analysis makes it possible to clarify general features of the problem itself, and to discuss some qualitative properties. Further accounts on spectral decomposition of the anisotropic elasticity tensor can be found in Sutcliffe (1992) and Theocaris and Sokolis (2000a,b). Banichuk deals with several problems, such as minimization of the compliance functional, the dynamic stiffness and the distortion energy. These problems are then generalized to the case of bodies consisting of several anisotropic phases; accordingly, the medium is represented as a polycrystalline aggregate.

The problem of extremizing the strain energy density by varying the mutual orientation of a fixed stress state to the material symmetry axes (regardless of the considered symmetry class) has also been developed by Cowin (1994). After showing that the stress and strain tensors commute at the stationarity (or critical) points of the strain energy, Cowin looks for absolute maxima and minima of the energy in a subset of orientations at which the gradient of the strain energy density vanishes respect to a second-order orthogonal tensor, representing the coordinate transformation. It is shown that 'the symmetry coordinate system of cubic symmetry is the only situation in linear anisotropic elasticity for which a strain energy density extremum can exist for all stress states'. The stationarity conditions for materials with other symmetries depend on the given stress state. In particular, the conditions for the energy extrema for transversely isotropic and orthotropic solids are found for uniaxial stress states. In Vianello (1996a) and Sgarra and

Vianello (1997a,b) attention is paid to showing the *existence* of rotations of the material axes with respect to the principal directions of strain, at which the energy density is stationary. By means of Weierstrass' theorem the existence of at least two such rotations is proved, which parametrically depend on the strain tensor for any material symmetry. At a first glance, this result seems to contradict the statement given in Cowin (1994); nevertheless, the difference with Cowin's formulation is that here the elastic symmetry is held fixed for a *specific* strain state, whereas in Cowin (1994) a *general* state is considered. This difference is exhaustively clarified in Cowin (1997). The extension to finite anisotropic elasticity is tackled by Blume (1994) and Vianello (1996b), where the properties of the extrema are shown to be the same as in the linear case. Further developments in this direction concern the problem of extremizing the strain energy density, with respect to both the orientation of the anisotropy axes and the type of material symmetry (Cowin and Yang, 2000), for a given, but arbitrary, stress state. This formulation reveals a strict connection with analogous problems concerning the generation of optimal topologies (Eschenauer and Olhoff, 2001), where it is essentially the microstructure of the solid that plays the role of design variable.

Finally, it is interesting to notice that the previously illustrated problems spontaneously arise not only in the study of the behaviour of man-made materials, but also in the mechanics of living tissues. For instance, Fyhrie and Carter (1986) develop a relationship between cancellous bone apparent density, trabecular orientation and applied stress, assuming the bone to be an orthotropic, self-optimizing material. It is shown that the trajectories of the material axes and the apparent density can be described by a unifying minimization principle involving a quadratic functional, similar to the strain energy density, and a purely quadratic Tsai-Wu failure criterion. The results predict the alignment of the material axes to the principal stress directions, in agreement with the previously reviewed results. Mechanisms of local changes in anisotropic properties, that more efficiently allow the living bone to carry the loads, are shown in Cowin (1987, 1995). These results suggest that the bone is designed by nature to have the greatest stiffness in axial direction and the greatest impact load resistance in the transverse one. The intimate relationship between trabecular architecture of cancellous bone and mechanics is also described by Odgaard et al. (1997).

3. Problem formulation

The problem of finding critical points of the strain energy density function, in linearly elastic anisotropic solids, is dealt with. In this problem, the local orientation of the anisotropy axes is assumed to be varying from a point to another through the body, and it is conceived as variable of the problem itself. The solid is supposed to be endowed with a positive definite strain energy. At first, no restriction on the type of elastic anisotropy is made. In an orthogonal reference system $z_1z_2z_3$, the constitutive law can be written in the form of the generalized Hooke's law:

$$T_{ij} = C_{ijhk}E_{hk}, \tag{1}$$

where T_{ij} and E_{hk} are the Cartesian components of the symmetric second-order stress and linearized strain tensors, respectively. C_{ijhk} are the components of the elasticity tensor of rank 4. From here onwards, summation over repeated indices (here ranging from 1 to 3) is understood. The type of anisotropy of the material is reflected by the symmetry group to which the elasticity tensor belongs (Smith and Rivlin, 1958; Gurtin, 1972). Symmetry of the strain and stress tensors, along with the postulated existence of an energy function, lead to the usual symmetries of the elasticity tensor:

$$C_{ijhk} = C_{jihk} = C_{ijkh} = C_{hkij}. \tag{2}$$

In the most general case, the elasticity tensor depends on 21 independent coefficients (triclinic system; Gurtin, 1972): this is the case of complete anisotropy, and no restriction is placed on the elasticities C_{ijhk} by any material symmetry property. Conversely, if the material possesses some planes or axes of elastic symmetry, the number of independent elastic coefficients is accordingly reduced. Constraints imposed by material symmetry on the elasticity tensor, classification of symmetry classes, and number of the different types of anisotropy, are topics widely discussed in the literature (see, among others, Love, 1994; Hearmon, 1961; Gurtin, 1972; Ting, 1996; Forte and Vianello, 1996; Huo and Del Piero, 1991; Cowin and Mehrabadi, 1995; Chadwick et al., 2001). For any material symmetry, it is customary to define, at each point P of the body, a 'principal', or 'material', orthogonal reference system $x_1x_2x_3$ in which the elasticity tensor shows the fewest number of independent non-vanishing components.

The relationship between the Cartesian components of the elasticity tensor in the global frame $z_1z_2z_3$, and those in the local material system $x_1x_2x_3$, denoted by $\widehat{C}_{mnpq}$, is given by the transformation law:

$$C_{ijhk} = Q_{im}Q_{jn}Q_{hp}Q_{kq}\widehat{C}_{mnpq}, \tag{3}$$

where Q_{ij} are the components of a proper orthogonal second-order tensor $\boldsymbol{Q}$.

The anisotropy of the solid is supposed to be *given*. The state of strain at each point P of the solid is characterized by the *given* values of the three principal strains and by the orthogonal principal strain directions $x_{\mathrm{I}}x_{\mathrm{II}}x_{\mathrm{III}}$.

Accordingly, at each point P of the solid *three* Cartesian orthogonal systems of axes are defined: $z_1z_2z_3$, parallel to the global system of coordinates, which form a set of axes common to all points in the body; $x_1x_2x_3$, aligned with the material axes, which can vary point by point; and $x_{\mathrm{I}}x_{\mathrm{II}}x_{\mathrm{III}}$, the system of the principal directions of strain.

When the material symmetry axes are locally rotated at any point in the body with respect to the fixed system $z_1z_2z_3$, the local orientations of the principal directions of strain change as well. Thus, any change in the energy density

$$W = \tfrac{1}{2}C_{ijhk}E_{ij}E_{hk} = \tfrac{1}{2}Q_{im}Q_{jn}Q_{hp}Q_{kq}\widehat{C}_{mnpq}E_{ij}E_{hk} \tag{4}$$

is due to a change in the mutual orientation between material axes $x_1x_2x_3$ and principal axes of strain $x_{\mathrm{I}}x_{\mathrm{II}}x_{\mathrm{III}}$. Accordingly, in Eq. (4) the Q_{ij} must be understood as components of a proper orthogonal tensor that rotates the material axes with respect to the principal directions of strain.

3.1. Formulation in the six-dimensional space

It is expedient to replace the three-dimensional formulation adopted so far with a suitable formulation of the constitutive law in the six-dimensional space. Different possible notational conventions can be found in the literature to express the stress–strain relationship (Walpole, 1984; Cowin and Mehrabadi, 1987; Mehrabadi and Cowin, 1990; Nadeau and Ferrari, 1998; Ting, 1996; Helnwein, 2001). Here, the description adopted is given by the following linear transformation is six dimensions (Walpole, 1984; Rychlewski, 1984; Cowin and Mehrabadi, 1992):

$$\boldsymbol{t} = \boldsymbol{C}\boldsymbol{e}, \tag{5}$$

where the two arrays $\boldsymbol{t}$ and $\boldsymbol{e}$ gather the six independent stress and strain components, respectively:

$$\boldsymbol{t} = (\,T_{11} \quad T_{22} \quad T_{33} \quad \sqrt{2}T_{23} \quad \sqrt{2}T_{31} \quad \sqrt{2}T_{12}\,)^{\mathrm{T}} \tag{6}$$

$$\boldsymbol{e} = (\,E_{11} \quad E_{22} \quad E_{33} \quad \sqrt{2}E_{23} \quad \sqrt{2}E_{31} \quad \sqrt{2}E_{12}\,)^{\mathrm{T}}. \tag{7}$$

The elasticity tensor is then consistently transformed into the 6×6 matrix, $\boldsymbol{C}$:

$$\boldsymbol{C}=\begin{pmatrix} C_{1111} & C_{1122} & C_{1133} & \sqrt{2}C_{1123} & \sqrt{2}C_{1131} & \sqrt{2}C_{1112} \\ C_{1122} & C_{2222} & C_{2233} & \sqrt{2}C_{2223} & \sqrt{2}C_{2231} & \sqrt{2}C_{2212} \\ C_{1133} & C_{2233} & C_{3333} & \sqrt{2}C_{3323} & \sqrt{2}C_{3331} & \sqrt{2}C_{3312} \\ \sqrt{2}C_{1123} & \sqrt{2}C_{2223} & \sqrt{2}C_{3323} & 2C_{2323} & 2C_{2331} & 2C_{2312} \\ \sqrt{2}C_{1131} & \sqrt{2}C_{2231} & \sqrt{2}C_{3331} & 2C_{2331} & 2C_{3131} & 2C_{3112} \\ \sqrt{2}C_{1112} & \sqrt{2}C_{2212} & \sqrt{2}C_{3312} & 2C_{2312} & 2C_{3112} & 2C_{1212} \end{pmatrix}. \tag{8}$$

According to this representation, the stress and strain tensors are mapped into the six-dimensional space in the same manner, contrary to the more frequently adopted Voigt's notation where only the shearing strains are affected by a multiplicative factor 2 (Love, 1994; Lekhnitskii, 1981; Sirotin and Chaskolkaïa, 1984; Mehrabadi and Cowin, 1990). The advantage of the Voigt's choice is that the components of the strain vector have the physical meaning of engineering strains. It has been proved by Mehrabadi and Cowin (1990) that the 6×6 matrix in (5) contains the components of a second-order tensor in six dimensions, whereas this tensorial character is lost in the Voigt's notation (Nye, 1957; Hearmon, 1961; Fedorov, 1968; Ting, 1996).

For the sake of conciseness, vector and tensor components in six dimensions will be denoted by lowercase letters, and the usual contraction of indices, which replaces any pair of indices with a single index (i.e., $11=1$, $22=2$, $33=3$, $23=32=4$, $31=13=5$ and $12=21=6$) is assumed. In such a way the matrix representation (5) can be explicitly written as

$$\begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \\ t_6 \end{pmatrix}=\begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{12} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{13} & c_{23} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{14} & c_{24} & c_{34} & c_{44} & c_{45} & c_{46} \\ c_{15} & c_{25} & c_{35} & c_{45} & c_{55} & c_{56} \\ c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & c_{66} \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{pmatrix}. \tag{9}$$

The components of the elasticity tensor in six-dimensions referred to the material symmetry axes will be denoted by $\hat{c}_{ij}$ $(i,j=1,\ldots,6)$ and collected into the matrix $\widehat{\boldsymbol{C}}$. To express the components of the second-rank elasticity tensor in any reference frame, c_{ij}, in terms of the elastic constants $\hat{c}_{ij}$, a suitable rotation tensor $\boldsymbol{q}$ in six dimensions must be defined, such that

$$c_{ij}=q_{im}q_{jn}\hat{c}_{mn}. \tag{10}$$

This equation represents the six-dimensional counterpart of Eq. (3). The definition of the orthogonal tensor $\boldsymbol{q}$ can be found in Mehrabadi and Cowin (1990), where its matrix representation is given as

$$\boldsymbol{q}=\begin{pmatrix} \boldsymbol{q}_{AA} & \boldsymbol{q}_{AB} \\ \boldsymbol{q}_{BA} & \boldsymbol{q}_{BB} \end{pmatrix} \tag{11}$$

with

$$\boldsymbol{q}_{AA}=\begin{pmatrix} Q_{11}^2 & Q_{12}^2 & Q_{13}^2 \\ Q_{21}^2 & Q_{22}^2 & Q_{23}^2 \\ Q_{31}^2 & Q_{32}^2 & Q_{33}^2 \end{pmatrix} \tag{12}$$

$$\boldsymbol{q}_{AB}=\begin{pmatrix} \sqrt{2}Q_{12}Q_{13} & \sqrt{2}Q_{13}Q_{11} & \sqrt{2}Q_{11}Q_{12} \\ \sqrt{2}Q_{22}Q_{23} & \sqrt{2}Q_{23}Q_{21} & \sqrt{2}Q_{21}Q_{22} \\ \sqrt{2}Q_{32}Q_{33} & \sqrt{2}Q_{33}Q_{31} & \sqrt{2}Q_{31}Q_{32} \end{pmatrix} \tag{13}$$

$$\boldsymbol{q}_{BA} = \begin{pmatrix} \sqrt{2}Q_{21}Q_{31} & \sqrt{2}Q_{22}Q_{32} & \sqrt{2}Q_{23}Q_{33} \\ \sqrt{2}Q_{31}Q_{11} & \sqrt{2}Q_{32}Q_{12} & \sqrt{2}Q_{33}Q_{13} \\ \sqrt{2}Q_{11}Q_{21} & \sqrt{2}Q_{12}Q_{22} & \sqrt{2}Q_{13}Q_{23} \end{pmatrix} \tag{14}$$

$$\boldsymbol{q}_{BB} = \begin{pmatrix} Q_{22}Q_{33} + Q_{23}Q_{32} & Q_{21}Q_{33} + Q_{23}Q_{31} & Q_{21}Q_{32} + Q_{22}Q_{31} \\ Q_{32}Q_{13} + Q_{33}Q_{12} & Q_{31}Q_{13} + Q_{33}Q_{11} & Q_{31}Q_{12} + Q_{32}Q_{11} \\ Q_{12}Q_{23} + Q_{13}Q_{22} & Q_{11}Q_{23} + Q_{13}Q_{21} & Q_{11}Q_{22} + Q_{12}Q_{21} \end{pmatrix}. \tag{15}$$

When the problem is written in the six-dimensional space, the energy density function (4) takes the form:

$$W = \tfrac{1}{2}c_{ij}e_i e_j = \tfrac{1}{2}q_{im}q_{jn}\hat{c}_{mn}e_i e_j \tag{16}$$

with $i, j = 1, 2, \ldots, 6$.

3.2. Condition for critical points of strain energy density

In this section the necessary condition for stationarity of the strain energy density is first briefly reviewed. This condition can be obtained in several ways (Seregin and Troitskii, 1981; Rovati and Taliercio, 1991, 1993; Cowin, 1994; Banichuk, 1996). Here it is preferred to recall the direct approach that makes use of the formulation in three dimensions (Cowin, 1994), where the physical meaning of the stationarity condition turns out in explicit form.

The objective stated in the previous section is to find stationarity points for the strain energy density function (4), according to the orthogonality constraint on tensor $\boldsymbol{Q}$, which, in terms of components, reads

$$Q_{ik}Q_{jk} = \delta_{ij} \tag{17}$$

where δ_{ij} is the Kronecker's delta. By means of the Lagrangian multipliers method, this constrained problem can be reformulated as an unconstrained one, consisting into the search for the stationarity of the augmented (or Lagrangian) function $\mathscr{L}$ (Cowin, 1994), defined as

$$\mathscr{L}(Q_{ij}; \Lambda_{ij}) = \tfrac{1}{2}C_{ijhk}E_{ij}E_{hk} - \Lambda_{ij}(Q_{ik}Q_{jk} - \delta_{ij}), \tag{18}$$

where Λ_{ij} are the components of a symmetric tensor $\boldsymbol{\Lambda}$ of rank 2. Stationarity of function $\mathscr{L}$ with respect to the Lagrangian multipliers Λ_{ij} restores the constraint (17), whereas stationarity with respect to variables Q_{ij}, that is, with respect to the local orientation of the anisotropy axes, is given by

$$\frac{\partial \mathscr{L}}{\partial Q_{rs}} = 2(\widehat{C}_{mspq}Q_{im}Q_{hp}Q_{kq}E_{ir}E_{hk} - \Lambda_{rj}Q_{js}) = 0, \tag{19}$$

where minor and major symmetries (2) of the elasticity tensor have been taken into account. After some algebraic manipulations, it is not difficult to show that

$$T_{ik}E_{ir} = \Lambda_{rk}, \tag{20}$$

which, by virtue of the symmetry of tensors $\boldsymbol{T}$, $\boldsymbol{E}$ and $\boldsymbol{\Lambda}$ allows one to write

$$\boldsymbol{TE} = \boldsymbol{ET}. \tag{21}$$

The commutativity of this product implies that the two tensors $\boldsymbol{T}$ and $\boldsymbol{E}$ are coaxial. Thus, the stationarity points of the strain energy density correspond to those orientations of the principal directions of strain to the material symmetry axes which make the principal directions of strain collinear with the principal directions of stress. Two second-order tensors are coaxial if they have a common triad of orthogonal eigenvectors. In isotropic elasticity, tensors $\boldsymbol{T}$ and $\boldsymbol{E}$ are always coaxial; this does not apply to anisotropic solids unless special conditions are fulfilled, which will be explicitly derived later for some classes of elastic symmetries.

This coaxiality requirement is the starting point for the solution procedure leading to the analytical determination of the orientation of the anisotropy axes to the principal directions of strain here proposed.

When the strain energy density is stationary, at each point P of the anisotropic body and in the Cartesian coordinate system $x_I x_{II} x_{III}$ of the principal directions of stress and strain, condition (21) implies

$$\begin{pmatrix} t_I \\ t_{II} \\ t_{III} \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{12} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{13} & c_{23} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{14} & c_{24} & c_{34} & c_{44} & c_{45} & c_{46} \\ c_{15} & c_{25} & c_{35} & c_{45} & c_{55} & c_{56} \\ c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & c_{66} \end{pmatrix} \begin{pmatrix} e_I \\ e_{II} \\ e_{III} \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{22}$$

which can be written, for notational purposes only, in concise form as

$$\begin{pmatrix} \boldsymbol{t}_p \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \boldsymbol{C}_{AA} & \boldsymbol{C}_{AB} \\ \boldsymbol{C}_{BA} & \boldsymbol{C}_{BB} \end{pmatrix} \begin{pmatrix} \boldsymbol{e}_p \\ \mathbf{0} \end{pmatrix} \tag{23}$$

(with $\boldsymbol{C}_{BA} = \boldsymbol{C}_{AB}^{T}$). Therefore, coaxiality of the stress and strain tensors can be expressed as

$$\boldsymbol{C}_{BA}\boldsymbol{e}_p = 0 \quad \Rightarrow \quad \begin{cases} c_{14}e_I + c_{24}e_{II} + c_{34}e_{III} = 0, \\ c_{15}e_I + c_{25}e_{II} + c_{35}e_{III} = 0, \\ c_{16}e_I + c_{26}e_{II} + c_{36}e_{III} = 0. \end{cases} \tag{24}$$

Clearly, system (24) is identically satisfied *for any* value of the principal strains e_I, e_{II}, e_{III} if all the coefficients $c_{14}, c_{24}, \ldots, c_{36}$ simultaneously vanish. This occurrence may happen only for those material symmetry classes for which at least a material coordinate system can be found where all the entries of submatrix $\boldsymbol{C}_{BA}$ vanish (Cowin, 1994, 1997), provided that, at the same time, these material axes are aligned with principal directions of stress and strain. These elastic symmetries correspond to the cubic system (characterized by 3 elastic coefficients), hexagonal(5) system (transverse isotropy, 5 coefficients), tetragonal(6) system (6 coefficients) and orthorombic symmetry (9 coefficients) (see Gurtin, 1972). For the other elastic symmetries, i.e. hexagonal (with 6 and 7 elastic coefficients), tetragonal (7 coefficients), monoclinic (13 coefficients) and triclinic (complete anisotropy, 21 coefficients), in any reference system the submatrix $\boldsymbol{C}_{BA}$ is different from the null matrix (Gurtin, 1972). Therefore, for such symmetries, no particular reference frame exists in which system (24) can be satisfied *for any* non-vanishing value of the principal strains. Eqs. (24) show that, for those elastic symmetries such that $\boldsymbol{C}_{BA} = \mathbf{0}$ in some coordinate system, stationarity of the energy can be achieved, in particular, for simultaneous coaxiality of principal directions of stress, strain and material symmetry axes. This is the special case considered by Cowin (1994). In the next sections it will be shown that coaxiality can be achieved under more general conditions.

It should be noticed that the necessary and sufficient condition under which the linear system (24) admits non-trivial solutions reads

$$\det \boldsymbol{C}_{BA} = \begin{vmatrix} c_{14} & c_{24} & c_{34} \\ c_{15} & c_{25} & c_{35} \\ c_{16} & c_{26} & c_{36} \end{vmatrix} = 0. \tag{25}$$

Thus, coaxiality of the stress and the strain tensors, and hence stationarity of the elastic energy density, is obtained when the equations of system (24) are linearly dependent.

If the Cartesian components of rotation are directly employed to describe the mutual orientation of the strain and the elasticity tensors, the orthogonality condition $\boldsymbol{Q}\boldsymbol{Q}^{T} = \mathbf{1}$ must be explicitly taken into account. This constraint makes computations heavy if closed form solutions are sought. Therefore it is preferable to assume as variables three independent unconstrained parameters, namely, three Euler's angles. The Euler's angles adopted here are visualized in Fig. 1. These angles characterize any finite rotation of x_I, x_{II}, x_{III} to

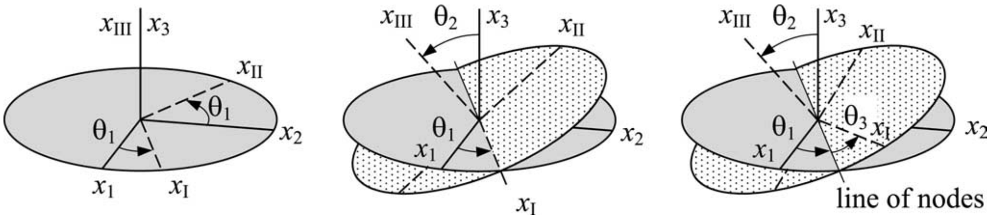

Fig. 1. Rotations defining the three Euler's angles.

x_1, x_2, x_3 as a sequence of three elementary rotations: the first one about x_3, by an angle θ_1, is followed by a rotation about x_{I} (in its new orientation) by an angle θ_2; the third rotation is about x_{III} (in its final orientation) by an angle θ_3 (Lurie, 2002).

Once the Euler's angles have been defined, the matrix representation of the proper orthogonal tensor $\boldsymbol{Q}$ reads

$$\begin{pmatrix} c_1c_3 - s_1c_2s_3 & s_1c_3 + c_1c_2s_3 & s_2s_3 \\ -c_1s_3 - s_1c_2c_3 & -s_1s_3 + c_1c_2c_3 & s_2c_3 \\ s_1s_2 & -c_1s_2 & c_2 \end{pmatrix}, \tag{26}$$

where the shorthand notations $s_i = \sin\theta_i$ and $c_i = \cos\theta_i$ ($i = 1, 2, 3$) have been adopted. It must be noticed that, for the purposes of this work, the directions in which the axes of the reference systems $x_{\mathrm{I}}x_{\mathrm{II}}x_{\mathrm{III}}$ and $x_1x_2x_3$ point is immaterial for the characterization of their relative orientation. Therefore, it is sufficient to allow the Euler's angles θ_1, θ_2, θ_3 to vary between 0 and π.

By expressing the elastic coefficients c_{ij} in (24) as functions of the Euler's angles θ_1, θ_2, θ_3 (through Eq. (10), definition (26) and the elements of tensor $\boldsymbol{q}$ given by Eqs. (12)–(15)), the condition $\det \boldsymbol{C}_{BA} = 0$ can be seen as a constraint on the values of the Euler's angles that allow the stress and strain tensors to be coaxial. The principal strains e_{I}, e_{II}, e_{III} compatible with such orientations can then be obtained as the eigensolutions of system (24) for any set of Euler's angles such that $\det \boldsymbol{C}_{BA} = 0$.

In this way, it is also possible to find those local orientations of the symmetry axes corresponding to critical values of the strain energy density, both *for any* strain state and for *particular* values of the principal strains.

Condition (25) can be rewritten in a slightly different form if one considers that any change in the strain energy density, associated with any rotation of the principal strain axes to the material axes, is due to the deviation from an isotropic term in the constitutive law of the material. From this point of view, the matrix $\widehat{\boldsymbol{C}}$ can be decomposed as the sum of an isotropic part $\widehat{\boldsymbol{I}}$ and an anisotropic part $\widehat{\boldsymbol{A}}$:

$$\widehat{\boldsymbol{C}} = \widehat{\boldsymbol{I}} + \widehat{\boldsymbol{A}}. \tag{27}$$

By means of the rotation matrix $\boldsymbol{q}$, and taking into account that the isotropic part is unaffected by rotations (i.e., $\boldsymbol{I} = \boldsymbol{q}\widehat{\boldsymbol{I}}\boldsymbol{q}^{\mathrm{T}} = \widehat{\boldsymbol{I}}$), the decomposition (27) can be rewritten in any reference frame as

$$\boldsymbol{C} = \boldsymbol{I} + \boldsymbol{q}\widehat{\boldsymbol{A}}\boldsymbol{q}^{\mathrm{T}} = \boldsymbol{I} + \boldsymbol{A}. \tag{28}$$

Consequently, it turns out that the strain energy density is the sum of an isotropic part W_I and an anisotropic contribution W_A, and reads

$$W = \tfrac{1}{2}(\boldsymbol{e} \cdot \boldsymbol{C}\boldsymbol{e}) = \tfrac{1}{2}[\boldsymbol{e} \cdot (\boldsymbol{I} + \boldsymbol{A})\boldsymbol{e}] = W_I + W_A, \tag{29}$$

where the dot denotes the usual inner product. Note that, in general, W_I and W_A cannot be individually interpreted as strain energy density functions: the decomposition (29) is introduced to confine in W_A the dependence of the strain energy density on the Euler's angles. The choice of the isotropic term $\widehat{\boldsymbol{I}}$ (and consequently, of the anisotropic part $\widehat{\boldsymbol{A}}$) in the decomposition (27) is somehow arbitrary. In any reference frame, the isotropic term adopted in the next sections is defined, on the basis of computational convenience, as

$$\boldsymbol{I} = \begin{pmatrix} \hat{c}_{13}+\hat{c}_{44} & \hat{c}_{13} & \hat{c}_{13} & 0 & 0 & 0 \\ \hat{c}_{13} & \hat{c}_{13}+\hat{c}_{44} & \hat{c}_{13} & 0 & 0 & 0 \\ \hat{c}_{13} & \hat{c}_{13} & \hat{c}_{13}+\hat{c}_{44} & 0 & 0 & 0 \\ 0 & 0 & 0 & \hat{c}_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \hat{c}_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \hat{c}_{44} \end{pmatrix}. \tag{30}$$

With this choice, the term W_I in (29) reads

$$W_I = \tfrac{1}{2}(\hat{c}_{13}\mathscr{I}_1^2 + \hat{c}_{44}\mathscr{I}_2) \tag{31}$$

with $\mathscr{I}_1 = \mathrm{tr}\,\boldsymbol{E}$ and $\mathscr{I}_2 = \mathrm{tr}\,\boldsymbol{E}^2$.

Finally, note that, by means of (27), condition (25) becomes $\det \boldsymbol{C}_{BA} = \det(\boldsymbol{I}_{BA} + \boldsymbol{A}_{BA}) = 0$, where the submatrices $\boldsymbol{I}_{BA}$ and $\boldsymbol{A}_{BA}$ are defined similarly to $\boldsymbol{C}_{BA}$ in (23), and $\boldsymbol{I}_{BA} = \boldsymbol{0}$ in any reference system. Thus, condition (25) can be rewritten as

$$\det \boldsymbol{A}_{BA} = \begin{vmatrix} a_{14} & a_{24} & a_{34} \\ a_{15} & a_{25} & a_{35} \\ a_{16} & a_{26} & a_{36} \end{vmatrix} = 0. \tag{32}$$

Condition (32) is then adopted in the next sections with reference to some classes of anisotropic solids, for which *all* the solutions in terms of Euler's angles are found explicitly. The corresponding values of the strain energy density are also computed and ordered.

4. Tetragonal symmetry

Solids pertaining to the tetragonal system are characterized by the existence of five planes of elastic mirror symmetry; the normals to four of the planes ($\boldsymbol{a}_i$, $i = 1, 2, 3, 4$) all lie in the fifth plane of symmetry, Π, normal to $\boldsymbol{a}_5$, and make angles of $\pi/4$ with respect to one another (Cowin and Mehrabadi, 1995). The normals to three of the planes of symmetry are coordinate axes, x_1, x_2, x_3 (see Fig. 2). In linear elasticity, the

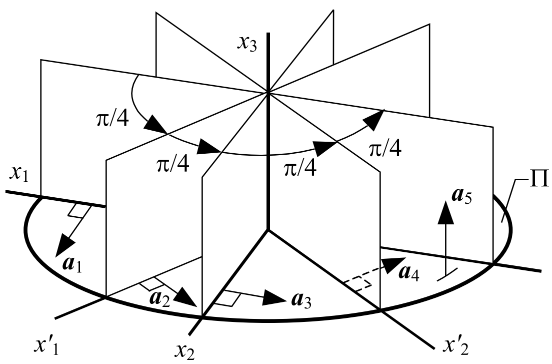

Fig. 2. Planes of elastic mirror symmetry for the tetragonal(6) system.

behaviour of these materials is defined by six independent stiffnesses. Referring to the coordinate system $x_1x_2x_3$, the stiffness matrix of the material reads

$$\widehat{\boldsymbol{C}} = \begin{pmatrix} \hat{c}_{11} & \hat{c}_{12} & \hat{c}_{13} & 0 & 0 & 0 \\ \hat{c}_{12} & \hat{c}_{11} & \hat{c}_{13} & 0 & 0 & 0 \\ \hat{c}_{13} & \hat{c}_{13} & \hat{c}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \hat{c}_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \hat{c}_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \hat{c}_{66} \end{pmatrix}. \tag{33}$$

In this class of material symmetry fall, as special cases, transversely isotropic materials (if $\hat{c}_{66} = \hat{c}_{11} - \hat{c}_{12}$), with five independent stiffnesses, and cubic materials (if $\hat{c}_{11} = \hat{c}_{33}$, $\hat{c}_{12} = \hat{c}_{13}$ and $\hat{c}_{44} = \hat{c}_{66}$), with three independent stiffnesses. These sub-cases will be dealt with in Sections 5 and 6, respectively.

Note that the material axes x_1 and x_2 are physically indistinguishable. In general, the elastic properties of a tetragonal solid in the planes of symmetry orthogonal to $\boldsymbol{a}_1$ and $\boldsymbol{a}_3$ (see Fig. 2) differ from those in the planes orthogonal to $\boldsymbol{a}_2$ and $\boldsymbol{a}_4$, except for the case of transversely isotropic materials. The intersections of the planes normal to $\boldsymbol{a}_2$ and $\boldsymbol{a}_4$ with the plane normal to $\boldsymbol{a}_5$ will be denoted by x_1' and x_2', and form another pair of physically indistinguishable material symmetry axes, different from x_1 and x_2.

According to Eqs. (27) and (30), the matrix $\widehat{\boldsymbol{A}}$ can be written as

$$\widehat{\boldsymbol{A}} = \begin{pmatrix} \hat{c}_{11} - \hat{c}_{13} - \hat{c}_{44} & \hat{c}_{12} - \hat{c}_{13} & 0 & 0 & 0 & 0 \\ \hat{c}_{12} - \hat{c}_{13} & \hat{c}_{11} - \hat{c}_{13} - \hat{c}_{44} & 0 & 0 & 0 & 0 \\ 0 & 0 & \hat{c}_{33} - \hat{c}_{13} - \hat{c}_{44} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \hat{c}_{66} - \hat{c}_{44} \end{pmatrix}$$

$$= \begin{pmatrix} \hat{a}_{11} & \hat{a}_{12} & 0 & 0 & 0 & 0 \\ \hat{a}_{12} & \hat{a}_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & \hat{a}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \hat{a}_{66} \end{pmatrix}. \tag{34}$$

Accordingly, the anisotropic contribution to the strain energy density W (see Eq. (29)) can be expressed as

$$W_A = \tfrac{1}{2}[\hat{a}_{11}(e_1^2 + e_2^2) + \hat{a}_{33}e_3^2 + 2\hat{a}_{12}e_1e_2 + \hat{a}_{66}e_6^2]. \tag{35}$$

4.1. Critical points of the strain energy density

To ensure coaxiality of the stress and the strain tensors at any point of a body with tetragonal elastic symmetry, the condition (32), i.e., $\det \boldsymbol{C}_{BA} = \det \boldsymbol{A}_{BA} = 0$, must be fulfilled, as pointed out in the preceding section. Explicitly, this conditions reads

$$-\frac{1}{4\sqrt{2}}(\hat{a}_{11} - \hat{a}_{12})(\hat{a}_{11} + \hat{a}_{12} - \hat{a}_{33})\hat{a}_{66}\sin^2\theta_2 \sin^2 2\theta_2 \sin^2 2\theta_3 = 0. \tag{36}$$

Apparently, from this equation it can be seen that particular materials exist, whose elastic constants are such that $\det(\boldsymbol{A}_{BA})$ vanishes for any orientation of the symmetry axes to the principal directions of strain. This is the case, for instance, with cubic materials: this particular situation will be carefully studied in Section 6. Thus, the necessary condition to achieve coaxiality of stresses and strains is that *at least* one of the Euler's angles θ_2 and θ_3 be equal either to 0 or $\pi/2$. This means that *at least* one of the principal directions of strain must lie in the plane x_1x_2: namely, this conditions applies for $x_{\rm I}$ if $\theta_3 = 0$, for $x_{\rm II}$ if $\theta_3 = \pi/2$, and for $x_{\rm III}$ if $\theta_2 = \pi/2$ (refer to Fig. 1). If $\theta_2 = 0$, both $x_{\rm I}$ and $x_{\rm II}$ lie in the plane x_1x_2.

Consider, for instance, the case where $\theta_3 = 0$. The sets of (linearly dependent) Eqs. (24), expressed in terms of coefficients $\hat{a}_{ij}$, which have to be fulfilled to ensure coaxiality of the stress and the strain tensors take the form:

$$\begin{cases} [2\alpha e_{\rm I} + (\beta - 4\hat{a}_{33})(e_{\rm II} + e_{\rm III}) + (\beta + 4\hat{a}_{33})(e_{\rm II} - e_{\rm III})\cos 2\theta_2 + \gamma \cos 4\theta_1 f(\theta_2)]\frac{\sin 2\theta_2}{8\sqrt{2}} = 0, \\ \gamma f(\theta_2)\frac{\sin 4\theta_1 \sin\theta_2}{4\sqrt{2}} = 0, \\ \gamma f(\theta_2)\frac{\sin 4\theta_1 \cos\theta_2}{4\sqrt{2}} = 0, \end{cases} \tag{37}$$

where the following material coefficients have been defined:

$$\alpha = \hat{a}_{11} + 3\hat{a}_{12} - \hat{a}_{66}, \tag{38}$$

$$\beta = 3\hat{a}_{11} + \hat{a}_{12} + \hat{a}_{66}, \tag{39}$$

$$\gamma = \hat{a}_{11} - \hat{a}_{12} - \hat{a}_{66}, \tag{40}$$

together with the function

$$f(\theta_2) = -2e_{\rm I} + e_{\rm II} + e_{\rm III} + (e_{\rm II} - e_{\rm III})\cos 2\theta_2. \tag{41}$$

It is worth noting that the material parameter γ can be expressed as

$$\gamma = 2(\hat{c}_{11} - c'_{11}), \tag{42}$$

where

$$c'_{11} = \tfrac{1}{2}(\hat{c}_{11} + \hat{c}_{12} + \hat{c}_{66}) \tag{43}$$

is the axial elasticity coefficient along x'_1. Thus, if $\gamma > 0$ (resp., $\gamma < 0$) the material is axially stiffer (resp., more flexible) along x_1 than along x'_1.

Assuming the principal strains to be all distinct, the system (37) can be fulfilled in the following cases:

(a) $\theta_1 = n\frac{\pi}{4}$ $(n = 0, 1, 2, 3)$ and $\theta_2 = 0$ or $\frac{\pi}{2}$.
(b) $\theta_1 = n\frac{\pi}{4}$ and θ_2 $(\neq 0, \pi/2)$ is such that the first equation in (37) is fulfilled, i.e., if n is even:

$$\cos 2\theta_2 = \frac{-2\hat{a}_{12}e_{\rm I} + (\hat{a}_{33} - \hat{a}_{11})(e_{\rm II} + e_{\rm III})}{\kappa(e_{\rm II} - e_{\rm III})} \tag{44}$$

with

$$\kappa = \hat{a}_{11} + \hat{a}_{33} = \hat{c}_{11} + \hat{c}_{33} - 2(\hat{c}_{13} + \hat{c}_{44}), \tag{45}$$

or, if n is odd:

$$\cos 2\theta_2 = \frac{-2a'_{12}e_{\rm I} + (a'_{33} - a'_{11})(e_{\rm II} + e_{\rm III})}{\kappa'(e_{\rm III} - e_{\rm II})} \tag{46}$$

with

$$\kappa' = a'_{11} + a'_{33} = \tfrac{1}{2}(\hat{c}_{11} + \hat{c}_{12} + \hat{c}_{66}) + \hat{c}_{33} - 2(\hat{c}_{13} + \hat{c}_{44}), \tag{47}$$

or alternatively

$$\kappa' = \kappa - \frac{\gamma}{2}. \tag{48}$$

The coefficients a'_{ij} are the components of $\boldsymbol{A}$ in the Cartesian reference frame $x'_1x'_2x_3$ rotated of $\pi/4$ about x_3 respect to the system $x_1x_2x_3$, and are given by

$$a'_{11} = \tfrac{1}{2}(\hat{a}_{11} + \hat{a}_{12} + \hat{a}_{66}), \quad a'_{12} = \tfrac{1}{2}(\hat{a}_{11} + \hat{a}_{12} - \hat{a}_{66}), \quad a'_{33} = \hat{a}_{33}. \tag{49}$$

(c) If θ_1 and θ_2 do not take any of the values listed in cases (a) and (b), the homogeneous system $\boldsymbol{A}_{BA}\boldsymbol{e}_{\mathrm{p}} = \boldsymbol{0}$ has rank 2: therefore, the principal strains for which collinearity of the stress and the strain tensors can be achieved are the eigensolutions of this system, and read

$$e_{\mathrm{II}} = \frac{1 - \delta\sin^2\theta_2}{\cos 2\theta_2} e_{\mathrm{I}}, \quad e_{\mathrm{III}} = -\frac{1 - \delta\cos^2\theta_2}{\cos 2\theta_2} e_{\mathrm{I}}, \tag{50}$$

with

$$\delta = 1 + \frac{\hat{a}_{11} + \hat{a}_{12}}{\hat{a}_{33}}. \tag{51}$$

These expressions can be easily combined to obtain a relationship involving the three principal strains, which ensures collinearity of the stress and the strain tensors, and reads

$$e_{\mathrm{II}} + e_{\mathrm{III}} = \delta e_{\mathrm{I}}. \tag{52}$$

When the principal strains fulfil this linear constraint, $\cos 2\theta_2$ can be expressed as

$$\cos 2\theta_2 = \frac{2e_{\mathrm{I}} - e_{\mathrm{II}} - e_{\mathrm{III}}}{e_{\mathrm{II}} - e_{\mathrm{III}}}, \tag{53}$$

so that $f(\theta_2) = 0$ and Eqs. (37) are fulfilled for any value of θ_1, provided that the angle defined by Eq. (53) exists, i.e., if

$$\left|\frac{e_{\mathrm{II}} + e_{\mathrm{III}}}{e_{\mathrm{II}} - e_{\mathrm{III}}}\right| \leqslant \left|\frac{\delta}{2 - \delta}\right|. \tag{54}$$

A similar discussion can be made when either x_{II} or x_{III} lies in the plane x_1x_2.

To summarize, three different situations can be encountered if the stress and the strain tensors are coaxial, corresponding to the cases listed above, namely:

(a) all the principal directions of strain are aligned with three of the normals to planes of material symmetry;
(b) one of the principal directions of strain is aligned with any one of the normals to the planes of material symmetry rotated of $\pi/4$ one to each other about x_3, $\boldsymbol{a}_i$, $i = 1, 2, 3, 4$ (i.e., is aligned with either one of the axes x_1, x'_1, x_2, x'_2);
(c) one of the principal directions of strain lies in the plane Π orthogonal to $\boldsymbol{a}_5$.

Note that a-type solutions are possible *for any* given state of strain, whereas b- and c-type solutions exist only if the principal strains fulfil certain constraints. Solutions of type b require the angle θ_2, defined by Eq. (44) or Eq. (46) (or by the corresponding angle that characterizes solutions with x_{II} or x_{III} lying in the plane x_1x_2) to exist: this is possible only in a certain region of the space of the principal strains $(e_{\mathrm{I}}, e_{\mathrm{II}}, e_{\mathrm{III}})$ which will be studied later. Solutions of type c exist only if the principal strains fulfil the linear constraint (52) (or the equivalent ones, that characterize the solutions with x_{II} or x_{III} lying in the plane x_1x_2).

Note also that, in turn, the cases a and b can be split into the following sub-cases:

(a′) and (b′), where at least one of the principal directions of strain is aligned with $\boldsymbol{a}_1$ or $\boldsymbol{a}_3$ (i.e., with x_1 or x_2);
(a″) and (b″), where at least one of the principal directions of strain is aligned with $\boldsymbol{a}_2$ or $\boldsymbol{a}_4$ (i.e., with x'_1 or x'_2).

Table 1
Euler's angles and orientations of the principal strain directions at the critical points of the strain energy density for tetragonal solids

Sol. type	x_{I}	x_{II}	x_{III}	θ_1	θ_2	θ_3
a'_1	x_3	x_1 or x_2	x_1 or x_2	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$
a'_2	x_1 or x_2	x_3	x_1 or x_2	0	$\frac{\pi}{2}$	0
a'_3	x_1 or x_2	x_1 or x_2	x_3	0	0	0
a''_1	x_3	x'_1 or x'_2	x'_1 or x'_2	$\frac{\pi}{4}$ or $\frac{3\pi}{4}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$
a''_2	x'_1 or x'_2	x_3	x'_1 or x'_2	$\frac{\pi}{4}$ or $3\frac{\pi}{4}$	$\frac{\pi}{2}$	0
a''_3	x'_1 or x'_2	x'_1 or x'_2	x_3	$-\theta_3+\frac{\pi}{4}$ or $+\frac{3\pi}{4}$	0	Any
b'_1	x_1 or x_2	$\in (x_1$ or $x_2, x_3)$	$\in (x_1$ or $x_2, x_3)$	0 or $\frac{\pi}{2}$	$\theta_{b'_1}$	0
b'_2	$\in (x_1$ or $x_2, x_3)$	x_1 or x_2	$\in (x_1$ or $x_2, x_3)$	0 or $\frac{\pi}{2}$	$\theta_{b''_2}$	$\frac{\pi}{2}$
b'_3	$\in (x_1$ or $x_2, x_3)$	$\in (x_1$ or $x_2, x_3)$	x_1 or x_2	0 or $\frac{\pi}{2}$	$\frac{\pi}{2}$	$\theta_{b'_3}$
b''_1	x'_1 or x'_2	$\in (x'_1$ or $x'_2, x_3)$	$\in (x'_1$ or $x'_2, x_3)$	$\frac{\pi}{4}$ or $\frac{3\pi}{4}$	$\theta_{b'_1}$	0
b''_2	$\in (x'_1$ or $x'_2, x_3)$	x'_1 or x'_2	$\in (x'_1$ or $x'_2, x_3)$	$\frac{\pi}{4}$ or $\frac{3\pi}{4}$	$\theta_{b''_2}$	$\frac{\pi}{2}$
b''_3	$\in (x'_1$ or $x'_2, x_3)$	$\in (x'_1$ or $x'_2, x_3)$	x'_1 or x'_2	$\frac{\pi}{4}$ or $\frac{3\pi}{4}$	$\frac{\pi}{2}$	$\theta_{b''_3}$
c_1	$\in (x_1, x_2)$	Any	Any	Any	θ_{c_1}	0
c_2	Any	$\in (x_1, x_2)$	Any	Any	θ_{c_2}	$\frac{\pi}{2}$
c_3	Any	Any	$\in (x_1, x_2)$	Any	$\frac{\pi}{2}$	θ_{c_3}

All the situations listed above are summarized in Table 1, where the values of the Euler's angles and the orientations of the principal strain directions to the material symmetry axes are given. Note that equivalent choices for the values of the Euler's angles other than those listed in Table 1, leading to the same physical orientations of the axes, can be made; here, they are disregarded for the sake of conciseness.

The values of the angles $\theta_{b'_i}$, $\theta_{b''_i}$, θ_{c_i} $(i = 1, 2, 3)$ in the last two columns of Table 1 will be explictly given later.

The solutions of type a', characterized by full collinearity of principal directions of strain and coordinate axes, are depicted in Fig. 3. The solutions of type a'' correspond to collinearity of two of the principal directions of strain with the material symmetry axes x'_1, x'_2 and are shown in Fig. 4.

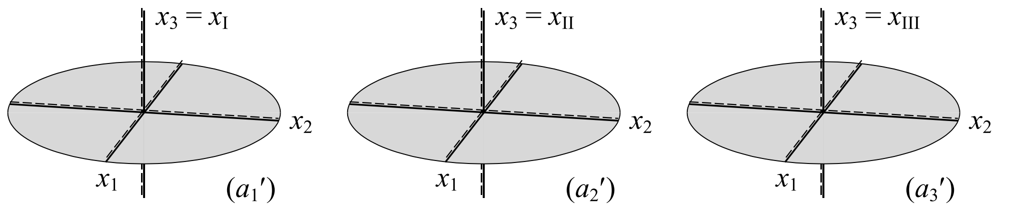

Fig. 3. Full collinearity of the principal directions of strain (dashed lines) and the coordinate axes (solid lines).

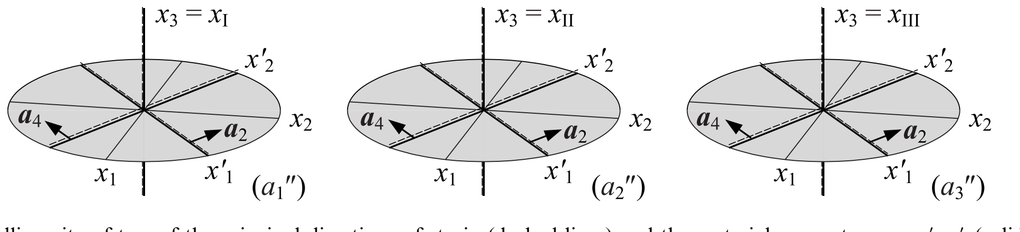

Fig. 4. Collinearity of two of the principal directions of strain (dashed lines) and the material symmetry axes x'_1, x'_2 (solid lines).

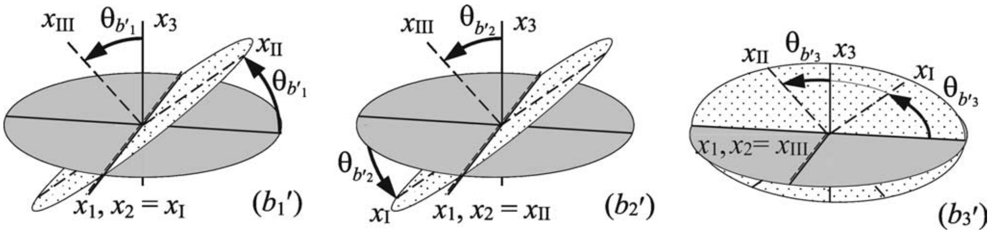

Fig. 5. Collinearity of one of the principal directions of strain (dashed lines) and one of the material symmetry axes (solid lines).

Fig. 5 shows the orientations of the principal directions of strain for solutions of type b'. These solutions are characterized by collinearity of only one of the principal directions of strain with either x_1 or x_2. The plane formed by the other two principal directions of strain contains the material axis x_3. The angles that one of these directions makes to x_3, indicated by $\theta_{b'_1}$, $\theta_{b'_2}$ and $\theta_{b'_3}$ in Table 1, are such that:

$$\cos 2\theta_{b'_1} = \frac{-2\hat{a}_{12}e_{\rm I} + (\hat{a}_{33} - \hat{a}_{11})(e_{\rm II} + e_{\rm III})}{\kappa(e_{\rm II} - e_{\rm III})}, \tag{55}$$

$$\cos 2\theta_{b'_2} = \frac{-2\hat{a}_{12}e_{\rm II} + (\hat{a}_{33} - \hat{a}_{11})(e_{\rm I} + e_{\rm III})}{\kappa(e_{\rm I} - e_{\rm III})}, \tag{56}$$

$$\cos 2\theta_{b'_3} = \frac{-2\hat{a}_{12}e_{\rm III} + (\hat{a}_{33} - \hat{a}_{11})(e_{\rm I} + e_{\rm II})}{\kappa(e_{\rm I} - e_{\rm II})}. \tag{57}$$

Whereas stationarity points of type a' and a'' for the strain energy density exist for any given strain state, stationarity points of type b' exist provided that the Euler's angles $\theta_{b'_i}$ ($i = 1, 2, 3$), given by Eqs. (55)–(57), can actually be defined. Their existence is conditioned by both the local state of strain and the elastic properties of the material. Referring, for the sake of illustration, to solution b'_1, the following inequalities must be fulfilled:

$$-1 \leqslant \frac{-2\hat{a}_{12}e_{\rm I} + (\hat{a}_{33} - \hat{a}_{11})(e_{\rm II} + e_{\rm III})}{\kappa(e_{\rm II} - e_{\rm III})} \leqslant 1. \tag{58}$$

These inequalities define a double-wedge shaped region in the plane $(e_{\rm II}/e_{\rm I}, e_{\rm III}/e_{\rm I})$, which is qualitatively plotted in Fig. 6, for a material with given elastic constants, together with the analogous admissible regions for cases b'_2 and b'_3. Note that the regions in which the three b'-type solutions are individually possible mutually intersect and do not cover the entire plane of normalized strains. This means that, according to the strain state, of the b'-type solutions either all can exist, or only some of them, or even none.

Solutions of type b'' are characterized by collinearity of one of the principal directions of strain with either x'_1 or x'_2 (see Fig. 7). The Euler's angles defining these situations ($\theta_{b''_i}$, $i = 1, 2, 3$, see Table 1) can be obtained from case b' by replacing the components $\hat{a}_{ij}$ with their homologous a'_{ij} in the reference frame $x'_1x'_2x_3$. Explicitly, these angles are such that:

$$\cos 2\theta_{b''_1} = \frac{-2a'_{12}e_{\rm I} + (a'_{33} - a'_{11})(e_{\rm II} + e_{\rm III})}{\kappa'(e_{\rm II} - e_{\rm III})}, \tag{59}$$

$$\cos 2\theta_{b''_2} = \frac{-2a'_{12}e_{\rm II} + (a'_{33} - a'_{11})(e_{\rm I} + e_{\rm III})}{\kappa'(e_{\rm I} - e_{\rm III})}, \tag{60}$$

$$\cos 2\theta_{b''_3} = \frac{-2a'_{12}e_{\rm III} + (a'_{33} - a'_{11})(e_{\rm I} + e_{\rm II})}{\kappa'(e_{\rm I} - e_{\rm II})}. \tag{61}$$

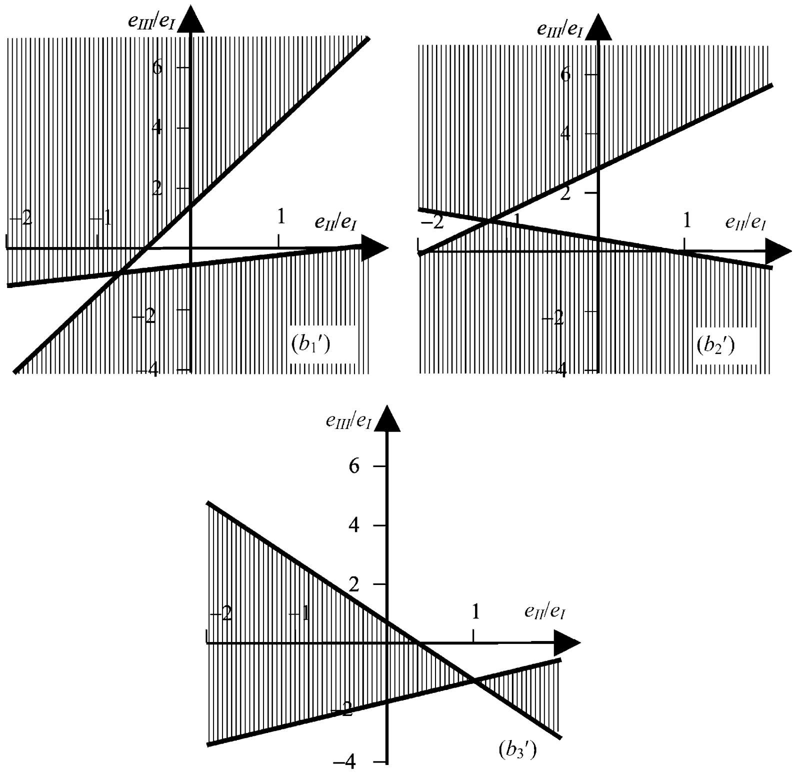

Fig. 6. Regions in the plane of the normalized principal strains $(e_{\mathrm{II}}/e_{\mathrm{I}}, e_{\mathrm{III}}/e_{\mathrm{I}})$ where b'-type solutions exist (dashed areas). Case of $Ca_2Sr(C_2H_5CO_2)_6$: $2\hat{a}_{12}/\kappa = -0.7405$, $(\hat{a}_{33} - \hat{a}_{11})/\kappa = 0.4747$.

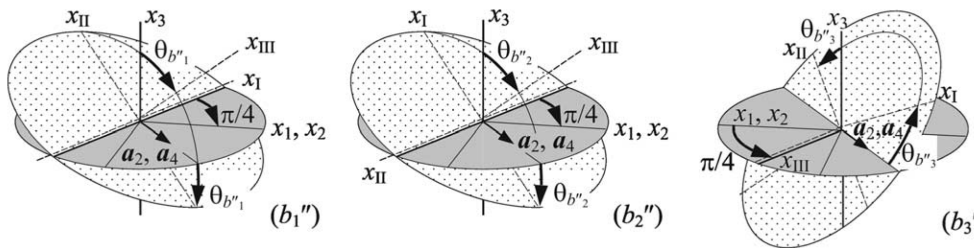

Fig. 7. Collinearity of one of the principal directions of strain (dashed lines) and one of the axes x_1', x_2' (solid lines).

The considerations previously made regarding the existence of the b'-type solutions apply also for b''-type solutions. These solutions exist provided that the point representative of the strain state, in the plane $(e_{\mathrm{II}}/e_{\mathrm{I}}, e_{\mathrm{III}}/e_{\mathrm{I}})$, falls within double-wedge shaped regions, similar to those shown in Fig. 6 for b'-type solutions. For example, the inequalities defining the region in which solution b_1'' exists can be obtained by replacing the $\hat{a}_{ij}$ with the homologous a_{ij}' in inequalities (58).

Finally, c-type solutions correspond to critical points of the strain energy density at which none of the principal strain directions is aligned with any one of the material symmetry axes, but one of these directions

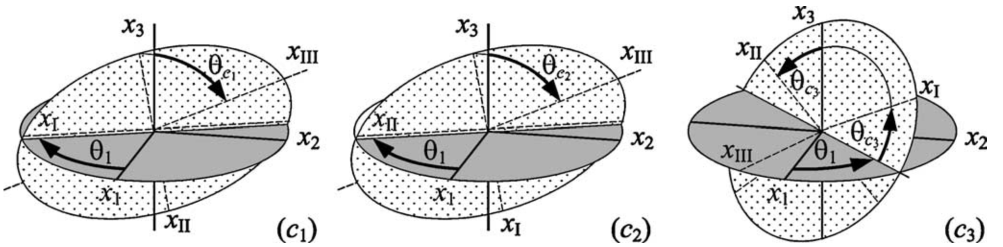

Fig. 8. Solutions characterized by one of the principal directions of strain (dashed lines) lying in the material symmetry plane (x_1, x_2), rotated of an arbitrary angle.

lies in the plane $\Pi = (x_1, x_2)$ (see Fig. 8). Provided that the principal strains fulfil particular linear constraints, the strain energy density turns out to be stationary *for any* value of θ_1. The values of the Euler's angles θ_{c_i}, $i = 1, 2, 3$ (see Table 1), are such that:

$$\cos 2\theta_{c_1} = \frac{2e_{\mathrm{I}} - e_{\mathrm{II}} - e_{\mathrm{III}}}{e_{\mathrm{II}} - e_{\mathrm{III}}} \quad \text{with } \delta e_{\mathrm{I}} = e_{\mathrm{II}} + e_{\mathrm{III}}, \tag{62}$$

$$\cos 2\theta_{c_2} = \frac{2e_{\mathrm{II}} - e_{\mathrm{III}} - e_{\mathrm{I}}}{e_{\mathrm{I}} - e_{\mathrm{III}}} \quad \text{with } \delta e_{\mathrm{II}} = e_{\mathrm{III}} + e_{\mathrm{I}}, \tag{63}$$

$$\cos 2\theta_{c_3} = \frac{2e_{\mathrm{III}} - e_{\mathrm{I}} - e_{\mathrm{II}}}{e_{\mathrm{I}} - e_{\mathrm{II}}} \quad \text{with } \delta e_{\mathrm{III}} = e_{\mathrm{I}} + e_{\mathrm{II}}. \tag{64}$$

Implicitly, the angles defined by Eqs. (62)–(64) are assumed to exist, i.e., the principal strains must fulfil inequalities similar to (54).

The linear constraints to be fulfilled by the principal strains in order to get c-type solutions can be represented by straight lines in the plane $(e_{\mathrm{II}}/e_{\mathrm{I}}, e_{\mathrm{III}}/e_{\mathrm{I}})$. For sake of illustration, in Fig. 9 the line corresponding to solution c_1 is plotted for a particular material, together with the regions where the b'_1- and b''_1-type solutions, with the same principal strain lying in the plane x_1x_2, exist.

4.2. Classification of the stationarity points

Once the stationarity points for the strain energy density have been identified, the relevant values of the energy are now computed and a classification of the stationarity points is made, according to the given values of the principal strains and the elastic constants.

The values of the strain energy density W at each critical point are listed in Table 2 in compact form, with $r = \mathrm{I, II, III}$ for $i = 1, 2, 3$, respectively, and $s, t \neq r$ subsequently taking the values I, II, III, with $s \neq t$.

The material parameters in Table 2, which characterize the values of the strain energy density at b'-type solutions are:

$$\eta_1 = \hat{c}_{11}[\hat{c}_{11} + \hat{c}_{33} - 2(\hat{c}_{13} + \hat{c}_{44})] - (\hat{c}_{12} - \hat{c}_{13})^2, \tag{65}$$

$$\eta_2 = \hat{c}_{11}\hat{c}_{33} - (\hat{c}_{13} + \hat{c}_{44})^2, \tag{66}$$

$$\eta_3 = \hat{c}_{11}\hat{c}_{13} + \hat{c}_{12}\hat{c}_{33} - (\hat{c}_{13} + \hat{c}_{44})(\hat{c}_{12} + \hat{c}_{13}), \tag{67}$$

$$\eta_4 = (\hat{c}_{33} - \hat{c}_{44})(\hat{c}_{11} - \hat{c}_{44}) - \hat{c}_{13}^2, \tag{68}$$

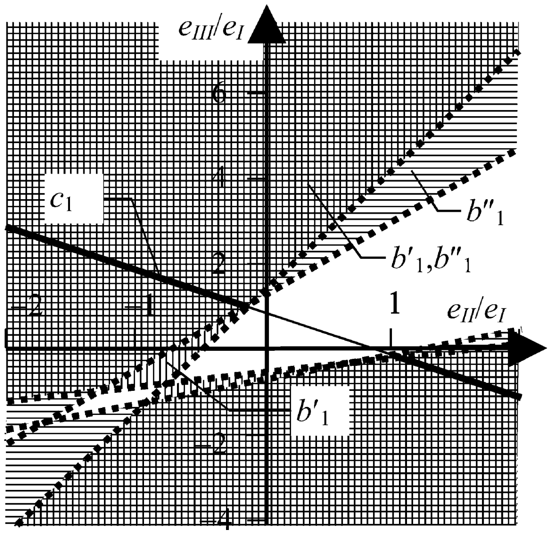

Fig. 9. Regions in the plane of the normalized principal strains $(e_{II}/e_I, e_{III}/e_I)$ where b'_1- and b''_1-type solutions exist and line corresponding to solution c_1. Case of $Ca_2Sr(C_2H_5CO_2)_6$: $2\hat{a}_{12}/\kappa = -0.7405$, $(\hat{a}_{33} - \hat{a}_{11})/\kappa = 0.4747$, $2\hat{a}'_{12}/\kappa' = -0.9185$, $(\hat{a}'_{33} - \hat{a}'_{11})/\kappa' = 0.2263$, $\delta = 0.8541$.

Table 2
Values of the strain energy density at the critical points for tetragonal solids

Sol. type	Strain energy density
a'	$W^{a'_i} = \frac{1}{2}[\hat{c}_{33}e_r^2 + \hat{c}_{11}(e_s^2 + e_t^2) + 2\hat{c}_{13}e_r(e_s + e_t) + 2\hat{c}_{12}e_se_t]$
a''	$W^{a''_i} = \frac{1}{4}[2\hat{c}_{33}e_r^2 + (\hat{c}_{11} + \hat{c}_{12})(e_s + e_t)^2 + \hat{c}_{66}(e_s - e_t)^2 + 4\hat{c}_{13}e_r(e_s + e_t)]$
b'	$W^{b'_i} = \frac{1}{2\kappa}[\eta_1 e_r^2 + \eta_2(e_s^2 + e_t^2) + 2\eta_3 e_r(e_s + e_t) + 2\eta_4 e_se_t]$
b''	$W^{b''_i} = \frac{1}{2\kappa'}[\eta'_1 e_r^2 + \eta'_2(e_s^2 + e_t^2) + 2\eta'_3 e_r(e_s + e_t) + 2\eta'_4 e_se_t]$

whereas the homologous parameters associated to b''-type solutions are:

$$\eta'_1 = (\hat{c}_{11} + \hat{c}_{12} + \hat{c}_{66})\left(\frac{\hat{c}_{33}}{2} - \hat{c}_{44}\right) + \hat{c}_{66}(\hat{c}_{11} + \hat{c}_{12} - 2\hat{c}_{13}) - \hat{c}_{13}^2, \tag{69}$$

$$\eta'_2 = \tfrac{1}{2}(\hat{c}_{11} + \hat{c}_{12} + \hat{c}_{66})\hat{c}_{33} - (\hat{c}_{13} + \hat{c}_{44})^2, \tag{70}$$

$$\eta'_3 = \tfrac{1}{2}(\hat{c}_{11} + \hat{c}_{12} - \hat{c}_{66})(\hat{c}_{33} - \hat{c}_{44}) - \hat{c}_{13}(\hat{c}_{13} + \hat{c}_{44} - \hat{c}_{66}), \tag{71}$$

$$\eta'_4 = (\hat{c}_{33} - \hat{c}_{44})(\tfrac{1}{2}(\hat{c}_{11} + \hat{c}_{12} + \hat{c}_{66}) - \hat{c}_{44}) - \hat{c}_{13}^2. \tag{72}$$

The value of the strain energy density corresponding to each c-type solution turns out to be numerically equal to that found for the homologous b'- and b''-type solutions with the same principal direction of strain lying in the plane Π, taking the constraints fulfilled by the principal strains into account (see Eqs. (62)–(64)). Explicitly,

$$W^{c_i} = W^{b'_i} = W^{b''_i}, \quad i = 1, 2, 3. \tag{73}$$

Thus, a maximum of twelve distinct values exists for the strain energy density at the stationarity points.

In order to classify the stationarity points, here it is proposed to compare the values of the strain energy density corresponding to solutions pertaining to each one of the classes a', a'', b' and b'', characterized by the same principal strain(s) lying in the plane Π. Compare first any pair of solutions belonging to classes a' and a'' with $x_{\rm III}$ in the plane Π, e.g., solutions a'_1 and a''_1 (but the same conclusions could be drawn by comparing solutions a'_2 and a''_2). Note that $x_{\rm I}$ is aligned with x_3 in both solutions. The difference between the corresponding values of the strain energy density is

$$W^{a'_1} - W^{a''_1} = \tfrac{1}{4}\gamma(e_{\rm II} - e_{\rm III})^2. \tag{74}$$

The sign of this difference depends uniquely on the material parameter γ. By generalizing this result, it is possible to state that, if $\gamma > 0$ one of the a'-type solutions has an energy higher than the energy associated with all of the a''-type solutions, and that one of the a''-type solutions has an energy lower than the energy associated with all of the a'-type solutions. The opposite holds for materials with $\gamma < 0$. Note that, if the state of strain and the material parameters are such that none of the solutions pertaining to classes $b'$and b'' exists, no further classification has to be done, and only the material parameter γ settles the class to which the stiffest solution belongs, and the class to which the most flexible one pertains.

Compare now the solutions belonging to classes a' and b' with $x_{\rm III}$ in the plane Π, for instance solutions a'_1 and b'_3, so that $x_{\rm III}$ is aligned with either x_1 or x_2 in both solutions. The difference between the corresponding values of the strain energy density is

$$W^{a'_1} - W^{b'_3} = \frac{1}{2\kappa}(\hat{a}_{11}e_{\rm I} + \hat{a}_{33}e_{\rm II} + \hat{a}_{12}e_{\rm III})^2. \tag{75}$$

The sign of this difference depends uniquely on the material parameter κ. By generalizing this result, it is possible to state that, if $\kappa > 0$ one of the a'-type solutions has an energy higher than the energy associated with all of the b'-type solutions, and that one of the b'-type solutions has an energy lower than the energy associated with all of the a'-type solutions. The opposite holds for materials with $\kappa < 0$.

Similar conclusions apply obviously when a''- and b''-type solutions are compared, with reference to the material parameter κ'.

Finally, consider a pair of solutions pertaining to classes b' and b'', with the same principal strain direction, say $x_{\rm III}$, aligned with either x_1 or x_2 in the former case, and with either x'_1 or x'_2 in the latter case. The difference between the corresponding values of the strain energy density is

$$W^{b'_3} - W^{b''_3} = \frac{\gamma}{4\kappa\kappa'}[(\hat{a}_{11} + \hat{a}_{12} + 2\hat{a}_{33})e_{\rm III} - \hat{a}_{33}\mathscr{I}_1]^2. \tag{76}$$

The sign of this difference depends only on the material parameter $\gamma/\kappa\kappa'$. It follows that, if $\gamma/\kappa\kappa' > 0$ (resp., $\gamma/\kappa\kappa' < 0$) one of the b'-type solutions has an energy higher (resp., lower) than the energy associated with all of the b''-type solutions, and that one of the b''-type solutions has an energy that is lower (resp., higher) than the energy at any one of the b'-type solutions.

When the energy values associated with the solutions pertaining to the remaining pairs of classes are compared, the signs of the relevant differences are explicitly affected by the values of the principal strains. Thus, no general conclusion can be drawn according uniquely to material parameters regarding the classification of the energy values relevant to solutions belonging to classes a' and b'', or to classes a'' and b'.

The classification of the stationarity points for the strain energy density is summarized in Fig. 10. Obviously, if any solution depending on the state of strain (b'- and b''-type solutions) is not admissible, the corresponding inequality in the chart of Fig. 10 has to be disregarded. By analyzing Fig. 10, for most materials with tetragonal symmetry it is possible to detect the class of solutions to which absolute maxima or absolute minima for the strain energy density belong, according only to the sign of the material parameters γ, κ and κ'. Exceptions are materials with γ, $\kappa > 0$ and $\kappa' < 0$, or with γ, $\kappa < 0$ and $\kappa' > 0$. In the former case, it is just possible to state that, a priori, absolute maxima pertain either to class a' or b'', and absolute minima pertain either to class a'' or b'; the opposite applies in the latter case. For these materials a complete classification requires the values of the principal strains to be explicitly taken into account, case by case.

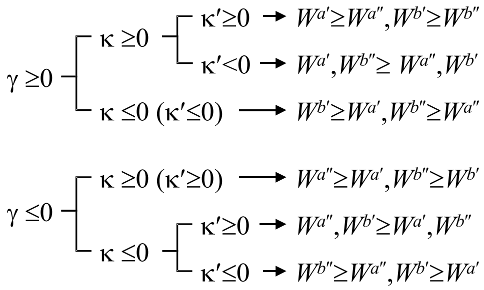

Fig. 10. Tetragonal symmetry: classification of the stationarity points for the strain energy density.

Once the classes of solutions that yield the extrema of the strain energy density have been identified, it is easy to mutually compare the three solutions pertaining to each class to detect which one corresponds to an absolute maximum or minimum.

For the sake of illustration, some plots of the strain energy density are now presented for a material with tetragonal symmetry, namely $Ba_2Si_2Ti_3O_8$, subjected to different states of strain. Two of the principal strains are kept constant in all of the cases considered (namely, $e_{II} = 10$, $e_{III} = -15$), whereas the remaining principal strain, e_I, takes different values in each case. The strain energy density W is normalized to the reference value W_I, and is plotted versus two of the Euler's angles, θ_1 and θ_2. The third angle, θ_3, is given a constant value of 0 (so that x_I lies in the plane x_1x_2 in all of the cases considered). The contour plots of the strain energy density are also shown, with the stationarity points for W marked out. The values of the elastic coefficients for the selected material, expressed in GPa, are (see Landolt and Börnstein, 1992): $\hat{c}_{11} = 140$, $\hat{c}_{33} = 83$, $\hat{c}_{44} = 66$, $\hat{c}_{66} = 128$, $\hat{c}_{12} = 36$, $\hat{c}_{13} = 24$, so that $\gamma = -14$, $\kappa = 43$ and $\kappa' = 50$.

Fig. 11(a) refers to the special case where the stationarity points corresponding to the solutions pertaining to all of the classes, a', a'', b', b'' and c, exist, (which, in the example, occurs at $e_I = 0.636$). In this case, the values of the strain energy density corresponding to solutions b'_1, b''_1, and c_1 all coincide: infinite stationarity points exist, which are independent on θ_1.

Note that, consistently with the chart in Fig. 10, one has

$$W^{a''_2} > W^{a'_2} > W^{a''_3} > W^{a'_3} > W^{b'_1} = W^{b''_1} = W^{c_1}. \tag{77}$$

Fig. 11(b) refers to any situation in which the values of the principal strains preclude the existence of c-type solutions: the value selected for e_I is 25. The angles θ_2 at which b'-type solutions exist are 0.934 and $\pi - 0.934$, whereas the angles at which b''_1-type solutions exist are 0.757 and $\pi - 0.757$. In this case, the values of the strain energy density corresponding to the stationarity points with $\theta_3 = 0$ are such that

$$W^{a''_2} > W^{a''_3} > W^{a'_3} > W^{a'_2} > W^{b''_1} > W^{b'_1}, \tag{78}$$

so that each of the solutions with at least one of the principal strain directions aligned with either x'_1 or x'_2 (i.e., a''_2, a''_3 or b''_1) has an energy higher than the homologous solution with the same principal direction(s) of strain aligned with either one of the coordinate axes x_1 or x_2 (i.e., a'_2, a'_3 or b'_1).

The state of strain to which Fig. 12(a) refers is such that both one of the b'-type solutions and the c-type solutions are missing; the value selected for e_I is 60. The angles θ_2 at which b''_1-type solutions exists are 0.898 and $\pi - 0.898$. In this case, the order for the values of the strain energy density at the stationarity points with $\theta_3 = 0$ is

$$W^{a''_3} > W^{a''_2} > W^{b''_1} > W^{a'_3} > W^{a'_2}, \tag{79}$$

which lends itself to the same remark made with reference to Fig. 11(b).

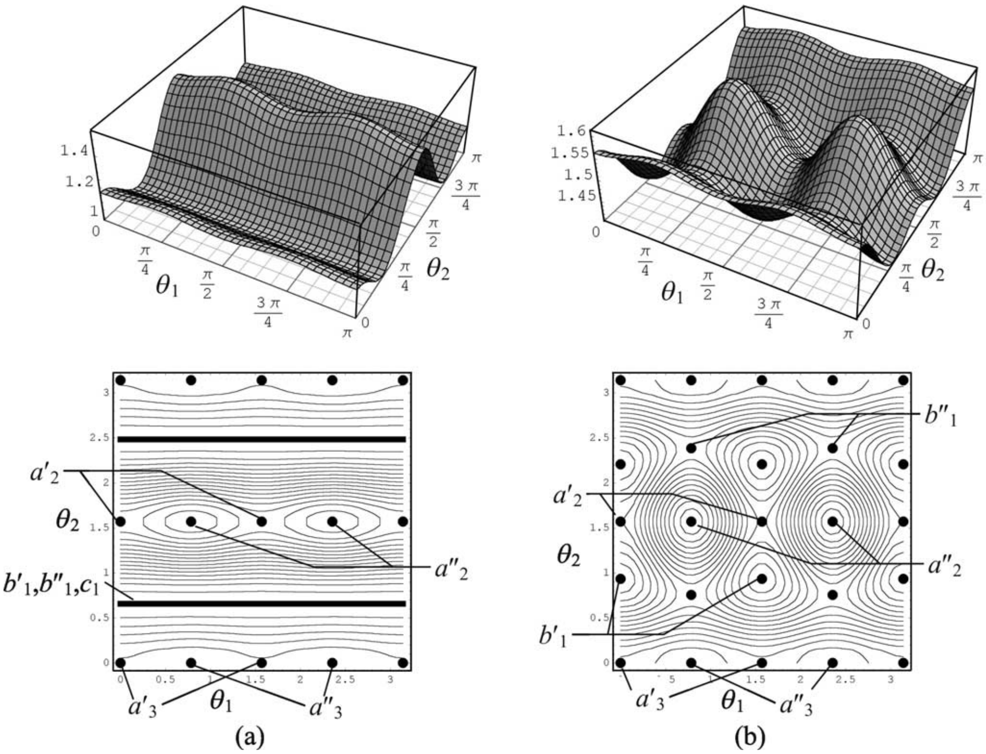

Fig. 11. Tetragonal symmetry: plots of the strain energy density for $Ba_2Si_2Ti_3O_8$ (normalized to W_I) versus the Euler's angles θ_1, θ_2 at $\theta_3 = 0$ and relevant contour plots—(a) existence of all types of solutions; (b) case in which *c*-type solutions do not exist.

Finally, Fig. 12(b) refers to a state of strain for which only a'- and a''-type solutions exist; the value selected for e_I is -100. The strain energy density takes higher values when the principal strains are aligned with $x'_1x'_2x_3$ rather than with the coordinate axes, namely,

$$W^{a''_2} > W^{a''_3} > W^{a'_2} > W^{a'_3}. \tag{80}$$

5. Transverse isotropy

The textured transversely isotropic symmetry is a special case of the crystalline hexagonal symmetry (see, e.g., Cowin and Mehrabadi, 1995). It is characterized by a plane of elastic mirror symmetry, Π, and an infinity of indistinguishable planes of mirror symmetry orthogonal to Π. All these planes intersect at the same axis, x_3, which turns out to be an axis of elastic symmetry of infinitely high order, i.e., an axis of rotational symmetry. Plane Π will be called 'plane of transverse isotropy'. This material symmetry can be seen as a special case of the tetragonal symmetry dealt with in the previous section and visualized in Fig. 2: Π is the plane normal to the unit vector $\boldsymbol{a}_5$, whereas any vector lying in Π is itself a normal to a plane of mirror symmetry. Examples of artificial transversely isotropic materials are, on the macroscopic scale, those materials having a bundled structure, as unidirectional

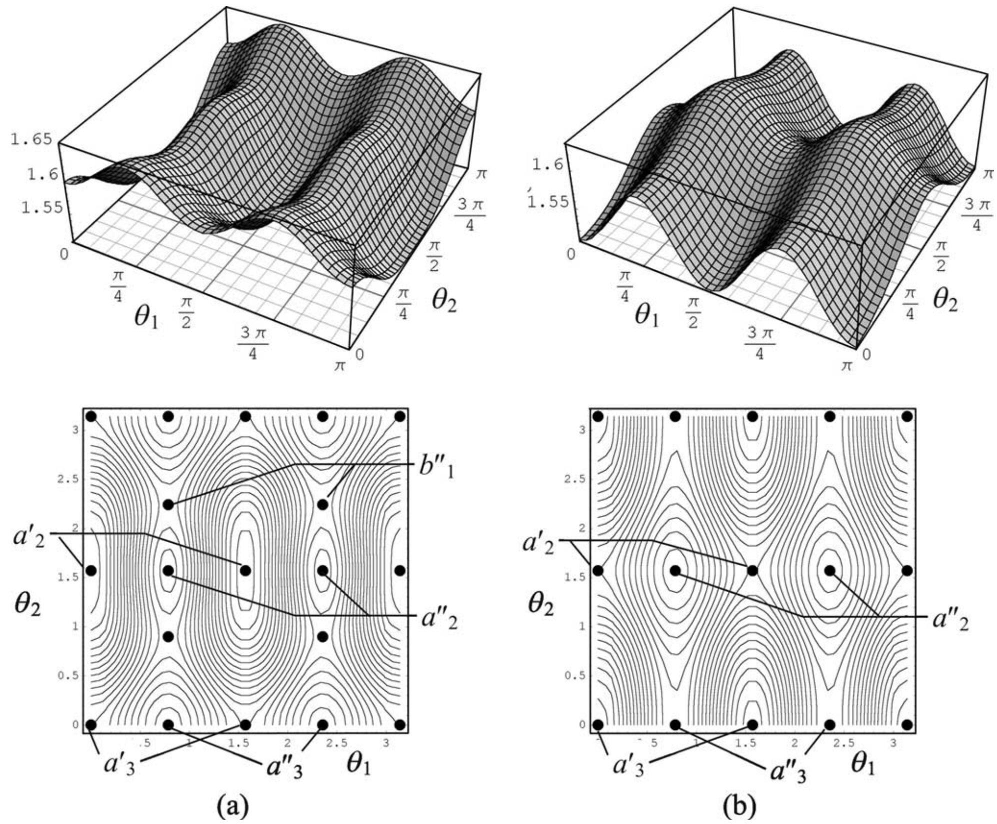

Fig. 12. Tetragonal symmetry: plots of the strain energy density for $Ba_2Si_2Ti_3O_8$ (normalized to W_I) versus the Euler's angles θ_1, θ_2 at $\theta_3 = 0$ and relevant contour plots—(a) case in which c- and b'-type solutions do not exist; (b) case in which only a-type solutions exist.

fiber reinforced composites, whereas layered rocks and soils, formed by the superposition of isotropic layers parallel to the bedding plane, are an example of natural macroscopically transversely isotropic media. Occasionally, the term 'cross-anisotropic' is found in the literature to denote transversely isotropic soil deposits (see, e.g., Bowles, 1988).

The linear elastic behaviour of transversely isotropic solids is defined by five independent elastic constants. Let $x_1x_2x_3$ be an orthogonal reference frame, with x_1 and x_2 being any pair of axes lying in the plane of transverse isotropy. In this reference frame, the matrix of the components of the elasticity tensor takes the form:

$$\widehat{\boldsymbol{C}} = \begin{pmatrix} \hat{c}_{11} & \hat{c}_{12} & \hat{c}_{13} & 0 & 0 & 0 \\ \hat{c}_{12} & \hat{c}_{11} & \hat{c}_{13} & 0 & 0 & 0 \\ \hat{c}_{13} & \hat{c}_{13} & \hat{c}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \hat{c}_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \hat{c}_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \hat{c}_{11} - \hat{c}_{12} \end{pmatrix}. \tag{81}$$

This is a special case of Eq. (33), where account is taken of the isotropic behaviour of the material in the plane Π, which implies $\hat{c}_{66} = \hat{c}_{11} - \hat{c}_{12}$. Note that the mutual orientation of any strain (or stress) tensor to the symmetry planes of a transversely isotropic solid is completely defined by the orientation of the axis of rotational symmetry, x_3, to the principal strain directions, $x_{\mathrm{I}}, x_{\mathrm{II}}, x_{\mathrm{III}}$. This orientation is known once two of the Euler's angles, θ_2 and θ_3, are, whereas θ_1 does not play any role.

By performing the decomposition of the elasticity tensor according to Eqs. (27) and (30), the strain energy density, W, can be expressed as the sum of an isotropic term W_I, given by Eq. (31), and an anisotropic term, W_A, given by

$$W_A = \tfrac{1}{2}[\hat{a}_{11}(e_1^2 + e_2^2 + e_6^2) + \hat{a}_{33}e_3^2 + 2\hat{a}_{12}(e_1 e_2 - \tfrac{1}{2}e_6^2)]. \tag{82}$$

Only this latter contribution is affected by relative rotations of the principal directions of strains to the material symmetry axes.

The necessary condition to be fulfilled by the strain tensor to achieve collinearity with the stress tensor, and thus stationarity of the strain energy density, is $\det \boldsymbol{C}_{BA} = \det \boldsymbol{A}_{BA} = 0$. Explicitly (see Eq. (36)),

$$-\frac{1}{4\sqrt{2}}(\hat{a}_{11} - \hat{a}_{12})^2(\hat{a}_{11} + \hat{a}_{12} - \hat{a}_{33})\sin^2\theta_2 \sin^2 2\theta_2 \sin^2 2\theta_3 = 0 \tag{83}$$

with $\hat{a}_{11} = \hat{c}_{11} - \hat{c}_{13} - \hat{c}_{44}$, $\hat{a}_{12} = \hat{c}_{12} - \hat{c}_{13}$ and $\hat{a}_{33} = \hat{c}_{33} - \hat{c}_{13} - \hat{c}_{44}$. This equations is fulfilled if at least one of the Euler's angles θ_2, θ_3 is either equal to zero or $\pi/2$, similarly to the more general case of solids with tetragonal symmetry. This condition amounts at requiring that *at least* one of the principal directions of strain must lie in the plane of transverse isotropy or that, alternatively, the axis of rotational symmetry must lie in any of the planes defined by a pair of principal directions of strain.

The classification of the possible orientations that ensure collinearity of the stress and the strain tensors, as well as the relevant energy values, can be deduced by the results established in the previous section. Since $\Pi = (x_1, x_2)$ is the plane of isotropy, only two classes of solutions exist, and are characterized by the following conditions:

(a) one of the principal directions of strain is aligned with the normal to the plane of transverse isotropy, i.e., with x_3; the other two lie in Π;
(b) only one of the principal directions of strain lies in the plane of transverse isotropy Π.

Both a'- and a''-type solutions for solids with tetragonal symmetry reduce to a single a-type class of solutions for transversely isotropic solids. Analogously, b'-, b''- and c-type solutions, obtained for the tetragonal case, all reduce to a single b-type class of solutions for transversely isotropic solids. Here again, a-type solutions are possible *for any* given state of strain, whereas b-type solutions exist only if the principal strains fulfil certain constraints, as discussed in Section 4.1.

The critical points for the strain energy density are summarized in Table 3, where the relevant values of the Euler's angles (θ_2, θ_3) and the orientations of the axis of rotational symmetry, x_3, to the principal directions of strain are listed. Note that different values can be given to the Euler's angles, other than those

Table 3
Euler's angles and orientations of the principal strain directions at the critical points for transversely isotropic solids

Sol. type	x_3	θ_2	θ_3
a_1	x_{I}	$\frac{\pi}{2}$	$\frac{\pi}{2}$
a_2	x_{II}	$\frac{\pi}{2}$	0
a_3	x_{III}	0	Any
b_1	$\in (x_{\mathrm{II}}, x_{\mathrm{III}})$	Eq. (55) or (59)	0
b_2	$\in (x_{\mathrm{I}}, x_{\mathrm{III}})$	Eq. (56) or (60)	$\frac{\pi}{2}$
b_3	$\in (x_{\mathrm{I}}, x_{\mathrm{II}})$	$\frac{\pi}{2}$	Eq. (57) or (61)

listed in Table 3, leading to the same physical orientations of the axes. The angles characterizing b-type solutions are given indifferently by the expressions obtained in Section 4.1 for b'- and b''-type solutions.

The values of the strain energy density corresponding to a- and b-type solutions are, respectively:

$$W^{a_i} = \frac{1}{2}(\hat{c}_{33}e_r^2 + \hat{c}_{11}(e_s^2 + e_t^2) + 2\hat{c}_{13}e_r(e_s + e_t) + 2\hat{c}_{12}e_s e_t), \tag{84}$$

$$W^{b_i} = \frac{1}{2\kappa}[\eta_1 e_r^2 + \eta_2(e_s^2 + e_t^2) + 2\eta_3 e_r(e_s + e_t) + 2\eta_4 e_r e_s]. \tag{85}$$

Here, $r = \mathrm{I}, \mathrm{II}, \mathrm{III}$ for $i = 1, 2, 3$, respectively, $s, t \neq r$ subsequently take the values I, II, III, with $s \neq t$, and η_j, $j = 1, \ldots, 4$, are given by Eqs. (65)–(68).

Stationarity points of type b exist provided that the Euler's angles θ_2 or θ_3 given by Eqs. (55)–(57) can actually be defined. Inequalities of the type (58) have then to be fulfilled, which involve both the principal strains and the elastic properties of the material.

5.1. Classification of the stationarity points

The problem of ordering the values corresponding to the different stationarity points for the strain energy density can be split into two separate sub-problems, similarly to the procedure followed in Section 4.2 for solids with tetragonal symmetry. First, the class of solutions (a or b) in which absolute maxima or minima fall are identified, according only to the sign of a material parameter. Then, the solutions corresponding to the extrema for the strain energy density are explicitly determined, by mutually comparing the three energy values pertaining to each class.

Compare a pair of solutions belonging to classes a and b, with the same principal strain direction (e.g., x_{III}) lying in the plane of transverse isotropy, Π. Provided that the Euler's angle which characterizes the b-type solution exists, the difference between the corresponding values of the strain energy density reads

$$W^{a_1} - W^{b_3} = \frac{1}{2\kappa}(\hat{a}_{11}e_{\mathrm{I}} + \hat{a}_{33}e_{\mathrm{II}} + \hat{a}_{12}e_{\mathrm{III}})^2. \tag{86}$$

The sign of this difference depends uniquely on the material parameter κ. Thus, similarly to what stated in Section 4.2, if $\kappa > 0$ one of the a-type (resp., b-type) solutions has an energy higher (resp., lower) than the energy associated with all of the b-type solutions. The opposite applies for materials with $\kappa < 0$. Accounting for this distinction, absolute maxima and minima can easily be obtained by exploring the three energy values pertaining to each class.

It is worth noting that the above classification is consistent with the chart shown in Fig. 10, taking into account that, for transversely isotropic materials, $\gamma = 0$ and $\kappa' = \kappa$.

For the sake of illustration, the strain energy density for a transversely isotropic solid is plotted in Fig. 13 versus two of the Euler's angles, θ_2 and θ_3. The value of the third angle, θ_1, is immaterial. The strain energy density W is normalized to the value W_I given by Eq. (31). The contour plots of the strain energy density are also shown, with the stationarity points for W marked out. The material selected is titanium boride (TiB_2): the values of the elastic coefficients for this material, expressed in GPa, are (see Landolt and Börnstein, 1992): $\hat{c}_{11} = 690$, $\hat{c}_{33} = 440$, $\hat{c}_{12} = 410$, $\hat{c}_{13} = 320$, $\hat{c}_{44} = 500$. Since for this material $\kappa = -510$ GPa, it is possible to state a priori that W finds its maximum at one of the b-type stationarity points (provided that at least one of these solutions exists) and its minimum at one of the a-type stationarity points. The values given to the principal strains (namely, $e_{\mathrm{I}} = 4$, $e_{\mathrm{II}} = -5$, $e_{\mathrm{III}} = 1$) are such that all of the three b-type solutions exist. The corresponding Euler's angles at which b-type solutions exist are: $\theta_2 = 0.7396$, $\pi - 0.7396$ and $\theta_3 = 0$ for solution b_1; $\theta_2 = 0.67$, $\pi - 0.67$ and $\theta_3 = \pi/2$ for solution b_2; $\theta_2 = \pi/2$, $\theta_3 = 0.793$, $\pi - 0.793$ for solution b_3. In this case, at the stationarity points the values of the strain energy density are such that

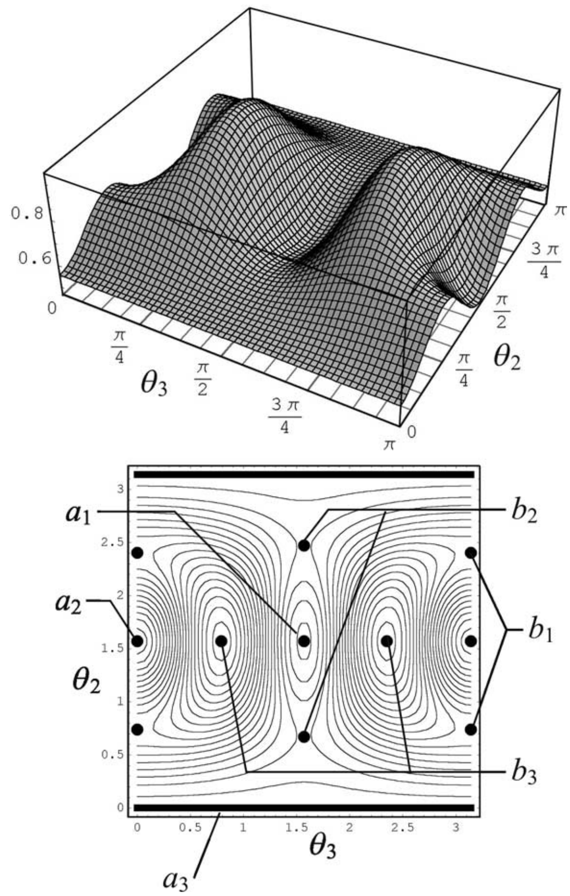

Fig. 13. Hexagonal(5) symmetry: plots of the strain energy density for TiB_2 (normalized to W_I) versus the Euler's angles θ_2, θ_3 and relevant contour plots.

$$W^{b_3} > W^{b_1} > W^{b_2} > W^{a_3} > W^{a_1} > W^{a_2}, \tag{87}$$

which is consistent with the negativity of κ.

6. Cubic symmetry

Cubic symmetry is characterized by nine planes of elastic mirror symmetry. This set of planes is formed by three mutually perpendicular planes, normal to the unit vectors $\boldsymbol{a}_1$, $\boldsymbol{a}_2$ and $\boldsymbol{a}_3$, and by six planes whose normals are $(\boldsymbol{a}_1 \pm \boldsymbol{a}_2)/\sqrt{2}$, $(\boldsymbol{a}_2 \pm \boldsymbol{a}_3)/\sqrt{2}$, and $(\boldsymbol{a}_3 \pm \boldsymbol{a}_1)/\sqrt{2}$ (see Fig. 14). The three planes of the former set are physically indistinguishable; the same applies for the six planes of the latter one, but the elastic properties exhibited by the material respect to any plane of the former set differ from those exhibited respect to any plane of the latter, unless the material is isotropic.

Let $x_1x_2x_3$ be a reference system collinear with $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$; the three axes are physically indistinguishable. In this frame, the matrix of the components of the elasticity tensor takes the form:

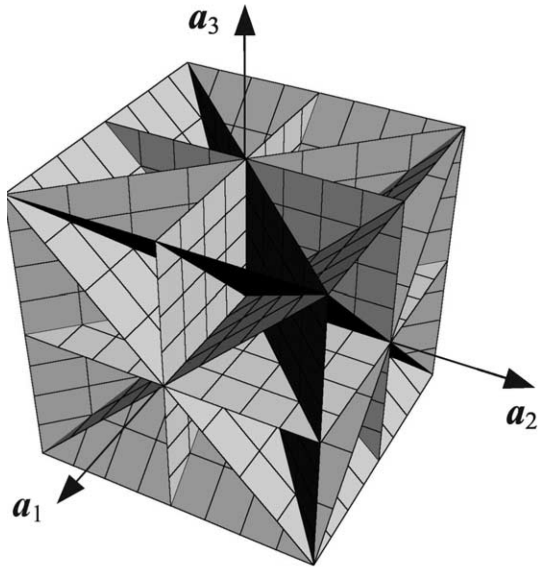

Fig. 14. Planes of elastic mirror symmetry for materials with cubic symmetry.

$$\widehat{\boldsymbol{C}} = \begin{pmatrix} \hat{c}_{11} & \hat{c}_{12} & \hat{c}_{12} & 0 & 0 & 0 \\ \hat{c}_{12} & \hat{c}_{11} & \hat{c}_{12} & 0 & 0 & 0 \\ \hat{c}_{12} & \hat{c}_{12} & \hat{c}_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \hat{c}_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \hat{c}_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \hat{c}_{44} \end{pmatrix}. \tag{88}$$

This is a special case of Eq. (33), with $\hat{c}_{33} = \hat{c}_{11}$, $\hat{c}_{13} = \hat{c}_{12}$, $\hat{c}_{66} = \hat{c}_{44}$.

In this case, the strain energy density is given by (see Eqs. (29), (31) and (35))

$$W = W_I + \tfrac{1}{2}\hat{a}_{11}(e_1^2 + e_2^2 + e_3^2), \tag{89}$$

with $\hat{a}_{11} = \hat{c}_{11} - \hat{c}_{12} - \hat{c}_{44} \equiv \gamma$, see Eq. (40).

Referring to Eq. (36), which is in general the necessary condition that ensures coaxiality of the stress and the strain tensors at any point of a body with tetragonal elastic symmetry, it immediately prompts out that this condition is identically fulfilled for materials with cubic symmetry. An alternative and independent proof of the identity $\det \boldsymbol{C}_{BA} = \det \boldsymbol{A}_{BA} \equiv 0$ for materials with cubic symmetry is given in Appendix. Thus, contrary to the cases of materials with tetragonal symmetry and transversely isotropic solids, where collinearity of the stress and the strain tensors can be achieved only provided that at least one of the principal directions of strain lies in the plane of material symmetry x_1x_2, no restriction of this type applies a priori for materials with cubic symmetry.

Since the matrix $\boldsymbol{C}_{BA}$ in Eqs. (24) is singular *for any strain state*, the search for the eigensolutions of this system proceeds differently from the case of tetragonal symmetry, where special values had to be imposed to either one of the Euler's angles θ_2 and θ_3 to achieve collinearity.

Unless the Euler's angles take special values, $\boldsymbol{C}_{BA}$ is of rank 2. Taking into account that $c_{43} = -(c_{41} + c_{42})$, $c_{53} = -(c_{51} + c_{52})$ and $c_{63} = -(c_{61} + c_{62})$ (see Appendix), the eigensolutions of the system (24) are $e_{\mathrm{I}} = e_{\mathrm{II}} = e_{\mathrm{III}}$. Thus, unless the state of strain is isotropic, the Euler's angles that render the strain energy density stationary must be such that the rank of $\boldsymbol{C}_{BA}$ is equal to one (or zero). By extracting any second-order minor of $\boldsymbol{C}_{BA}$ (e.g., the algebraic complement of c_{63}, Δ_{63}), one has

$$\begin{aligned}\Delta_{63} &= c_{41}c_{52} - c_{42}c_{51}\\ &= \tfrac{1}{2}\hat{a}_{11}^2 \sin 2\theta_1 \sin 2\theta_2 \sin\theta_2[\cos 2\theta_1 \cos 2\theta_2 \cos 2\theta_3 - \tfrac{1}{4}\sin 2\theta_1 \cos\theta_2(1+3\cos 2\theta_2)\sin 2\theta_3],\end{aligned} \tag{90}$$

which vanishes if either θ_1 or θ_2 is equal to 0 or $\pi/2$, or if the Euler's angles are such that

$$\tan 2\theta_1 \tan 2\theta_3 = \frac{4\cos 2\theta_2}{\cos\theta_2(1+3\cos 2\theta_2)}. \tag{91}$$

It is possible to show that any other minor of $\boldsymbol{C}_{BA}$ vanishes if Eq. (91) applies; provided that the Euler's angles fulfil Eq. (91), the rank of $\boldsymbol{C}_{BA}$ is equal to one. By inspection of the other minors of order two of $\boldsymbol{C}_{BA}$, one finds that other cases in which this matrix is at most of rank 1 are:

$$\theta_1 = 0 \text{ or } \frac{\pi}{2}, \quad \theta_2 \text{ (or } \theta_3) = \frac{\pi}{4} \text{ or } \frac{3\pi}{4}, \tag{92}$$

$$\theta_2 = \pm\frac{\pi}{2}, \quad \theta_1 \text{ (or } \theta_3) = \frac{\pi}{4} \text{ or } \frac{3\pi}{4}, \tag{93}$$

$$\theta_2 = 0 \quad \forall\theta_1, \theta_3. \tag{94}$$

Some of these combinations of Euler's angles also fulfil Eq. (91).

Once the conditions ensuring the possibility for the strain energy density to be stationary for non-isotropic states of strain have been singled out, the search for the stationarity points proceeds similarly to the procedure followed in Section 4. The combinations of Euler's angles that make vanish all the minors of order two of $\boldsymbol{C}_{BA}$ are subsequently substituted in Eq. (24), and the system is found to be fulfilled either:

(a) by special sets of angles that make the matrix of rank 0, regardless of the state of strain, or
(b) by combinations of the principal strains dependent on the Euler's angles, if $\boldsymbol{C}_{BA}$ is of rank 1.

The former possibility, (a), corresponds to solutions referred to as a' and a'' in Section 4: either all the principal strains are aligned with the material symmetry axes x_1, x_2, x_3 (a'-type solutions), or one of the principal strains is aligned with one of the material symmetry axes, with the other two rotated of $\pi/4$ to the remaining symmetry axes (a''-type solutions). Note that b'-type solutions found in Section 4.1 reduce to a''-type solutions for cubic materials. Being the material axes mutually interchangeable, the strain energy density takes the same value at any one of the a'-type solutions, that is (see Eq. (89))

$$W^{a'} = \tfrac{1}{2}(\hat{c}_{12}\mathscr{I}_1^2 + (\hat{c}_{11} - \hat{c}_{12})\mathscr{I}_2). \tag{95}$$

The value of the strain energy density at any a''-type solution, with any principal strain direction x_r aligned with any one of the material symmetry axes $x_1x_2x_3$, is

$$W^{a''_i} = \tfrac{1}{2}(\hat{c}_{12}\mathscr{I}_1^2 + \hat{c}_{44}\mathscr{I}_2 + (\hat{c}_{11} - \hat{c}_{12} - \hat{c}_{44})(e_r^2 + \tfrac{1}{2}(\mathscr{I}_1 - e_r)^2)) \tag{96}$$

with $r = \mathrm{I}, \mathrm{II}, \mathrm{III}$ for $i = 1, 2, 3$, respectively.

The latter possibility, (b), corresponds to combinations of the Euler's angles that fulfil Eq. (91). Note that, in particular, Eq. (91) is satisfied whenever θ_1 (or θ_3) is equal to $n\pi/2$ (n integer) and $\theta_2 = m\pi/4$ (m odd), etc. These cases generalize the solutions referred to as b'' in Section 4, with one of the principal directions of strain aligned with any bisector of a couple of coordinate axes. However, contrary to materials with tetragonal symmetry or transversely isotropic solids, for cubic symmetry the stress and strain tensors can be coaxial also when none of the principal directions of strain lies in any one of the material symmetry planes, provided that Eq. (91) applies. In this instance, solving any one of the three equations forming the system (24) for one of the principal strains (say, $e_{\rm III}$), one gets

$$e_{\rm III} = \frac{e_{\rm I}+e_{\rm II}}{2} - \frac{3}{2}(e_{\rm I}-e_{\rm II})\cos 2\theta_3 \frac{1-\cos 2\theta_2}{1+3\cos 2\theta_2}. \tag{97}$$

Rearranging Eqs. (91) and (97), it is possible to express the values of two of the Euler's angles, at which the stress and the strain tensor are collinear, in terms of the third angle and the given principal strains:

$$\cos 2\theta_2 = 1 - \frac{2}{3}\frac{e_{\rm I}+e_{\rm II}-2e_{\rm III}}{e_{\rm I}\sin^2\theta_3 + e_{\rm II}\cos^2\theta_3 - e_{\rm III}} \tag{98}$$

$$\tan 2\theta_1 = \frac{e_{\rm I}\cos^2\theta_3 + e_{\rm II}\sin^2\theta_3 - (e_{\rm I}\sin^2\theta_3 + e_{\rm II}\cos^2\theta_3)\cos^2\theta_2 - e_{\rm III}\sin^2\theta_2}{(e_{\rm I}-e_{\rm II})\cos\theta_2\sin 2\theta_3}. \tag{99}$$

Eq. (98) shows that, if

$$\left|\frac{e_{\rm I}-e_{\rm II}}{e_{\rm I}+e_{\rm II}-2e_{\rm III}}\right| \leqslant \tfrac{1}{3}, \tag{100}$$

the Euler's angle θ_2 defined by Eq. (98) exists for any value of θ_3, and so does θ_1 defined by Eq. (99). If the principal strains do not fulfil the constraint (100), θ_3 cannot take arbitrary values; note, however, that for any given strain state it is always possible to find values of θ_3 that give real values for θ_2 according to Eq. (98), that is, b-type critical points.

It is interesting to note that, by computing the axial strain components along the coordinate axes x_1, x_2, x_3 accounting for Eqs. (98) and (99), one gets

$$e_1 = e_2 = e_3 = \tfrac{1}{3}\mathscr{I}_1, \tag{101}$$

which means that b-type solutions are characterized by equally strained material symmetry axes. Taking Eq. (101) into account, from Eq. (89) it readily prompts out that the value of the strain energy density in any b-type solution is

$$W^b = W_I + \tfrac{1}{6}\hat{a}_{11}\tfrac{1}{2}\mathscr{I}_1^2 = \tfrac{1}{2}[\tfrac{1}{3}(\hat{c}_{11}+2\hat{c}_{12}-\hat{c}_{44})\mathscr{I}_1^2 + \hat{c}_{44}\mathscr{I}_2]. \tag{102}$$

To summarize, it is possible to state that, for materials with cubic symmetry, the strain energy density is stationary respect to the Euler's angles either if at least one of the principal strain directions is aligned with one of the material symmetry axes (and the other two are collinear with the remaining symmetry axes, or bisect them), or if the principal strains are rotated to the material symmetry axes so as to make equal the axial strains along them. These results were already established in Rovati and Taliercio (1991) through an alternative approach.

6.1. Classification of the stationarity points

To order the values of the strain energy density for a cubic solid at the critical points, the values corresponding to any pair of stationarity points are subsequently compared. Consider first a'- and b-type solutions and subtract Eq. (102) from Eq. (95):

$$W^{a'} - W^b = \tfrac{1}{2}\hat{a}_{11}(\mathscr{I}_2 - \tfrac{1}{3}\mathscr{I}_1^2) = \tfrac{1}{6}\hat{a}_{11}((e_{\rm I}-e_{\rm II})^2 + (e_{\rm II}-e_{\rm III})^2 + (e_{\rm III}-e_{\rm I})^2). \tag{103}$$

Thus, $W^{a'} \geqslant W^b$ for materials with $\hat{a}_{11} > 0$, whereas $W^{a'} \leqslant W^b$ for materials with $\hat{a}_{11} < 0$.

As far as the comparison between a'- and a''-type solutions is concerned, referring to the results established in Section 4.2 for materials with tetragonal symmetry, it is possible to state that $W^{a'} - W^{a''_i} \geqslant 0$ (resp. $\leqslant 0$) for any i, if $\gamma(\equiv \hat{a}_{11}) > 0$ (resp. <0).

Consider now the difference of the strain energy density associated with any a''-type solution (Eq. (96)) and with b-type solutions (Eq. (102)):

$$W^{a''_i} - W^b = \frac{1}{2}\hat{a}_{11}\left(e_r^2 + \frac{1}{2}(\mathscr{I}_1 - e_r)^2\right) - \frac{1}{6}\hat{a}_{11}\frac{1}{2}\mathscr{I}_1^2 = \frac{1}{12}\hat{a}_{11}(3e_r - \mathscr{I}_1)^2, \tag{104}$$

with $r = \mathrm{I}, \mathrm{II}, \mathrm{III}$ for $i = 1, 2, 3$, respectively. Thus, $W^{a''_i} - W^b \geqslant 0$ (resp. $\leqslant 0$) for any i, if $\gamma(\equiv \hat{a}_{11}) > 0$ (resp. <0).

To summarize, it is possible to state that the extrema for the strain energy density of solids with cubic symmetry always correspond to a'-type or to b-type solutions, that is, to full collinearity or no collinearity of the material axes x_1, x_2, x_3 with the principal directions of strain, respectively. $W^{a'}$ is an absolute maximum and W^b is an absolute minimum for materials with $\hat{a}_{11} > 0$; the opposite applies for materials with $\hat{a}_{11} < 0$. Solutions of a''-type, with only one of the principal strain directions aligned with any one of the material symmetry axes, are just relative maxima or minima.

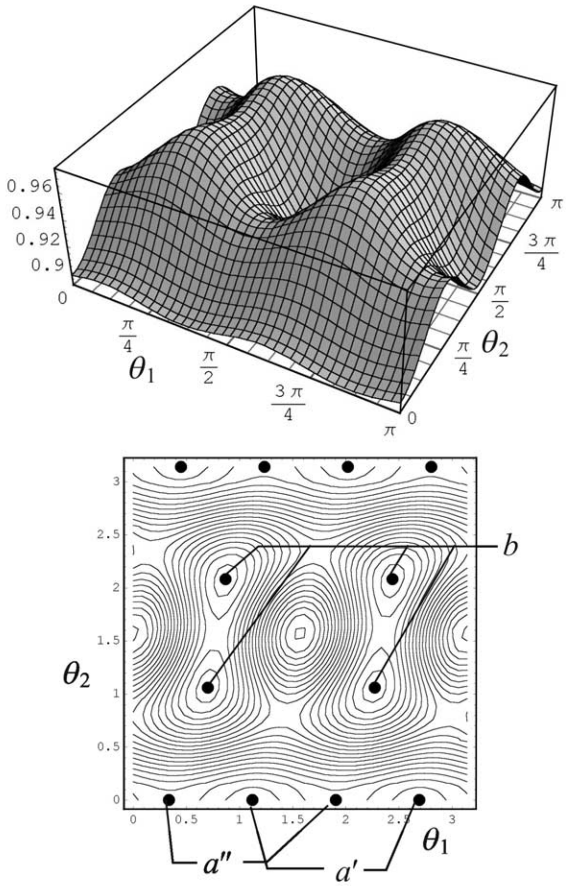

Fig. 15. Cubic symmetry: plots of the strain energy density for aluminium (normalized to W_I) versus the Euler's angles θ_1, θ_2 at $\theta_3 = \pi/7$ and relevant contour plots.

For the sake of illustration, in Fig. 15 the strain energy density of aluminum (Al) subjected to a state of strain characterized by $e_{\rm I} = 10$, $e_{\rm II} = 5$, $e_{\rm III} = -5$, is plotted versus two of the Euler's angles, θ_1 and θ_2. The third angle, θ_3, is given a constant value of $\pi/7$. The elastic constants of this cubic material (in GPa, see Landolt and Börnstein, 1992) are $\hat{c}_{11} = 108$, $\hat{c}_{12} = 62$, $\hat{c}_{44} = 56.6$, so that $\hat{a}_{11} = -10.6 < 0$. The strain energy density is normalized to W_I, given by Eq. (31). With the values chosen for the principal strains, inequality (100) applies, and b-type solutions exist for any θ_3. The contour plots of the strain energy density are also reported, with the points corresponding to any stationarity point marked out. Note that some points in the surface representing W/W_I correspond to partial stationarity with respect to θ_1 and θ_2, but not with respect to θ_3.

By direct inspection of Fig. 15, one can immediately note that

$$W^b > W^{a_3''} > W^{a'}, \tag{105}$$

which is consistent with $\hat{a}_{11} < 0$.

7. Concluding remarks

With reference to linearly elastic anisotropic solids, the orientations of the material symmetry axes to the principal directions of strain (or stress) corresponding to critical points for the strain energy density function have been sought. First, the property of coaxiality that the stress and strain tensors meet when these critical points are attained has been outlined. Then, on the basis of such a general property, *all* the orientations corresponding to absolute and relative maxima and minima have been found for tetragonal (6), hexagonal (5) and cubic material symmetries, in terms of triplets of Euler's angles. For each of these symmetry classes, the corresponding stationarity values of the energy density have been analytically computed and compared each other.

In the case of tetragonal symmetry (Section 4), three classes of solutions have been found. The first class is that identified by Cowin (1994), and is characterized by complete collinearity of the principal directions of stress, strain, and material symmetry axes. In this class, stationarity points of the strain energy density exist for any strain state. The second one is characterized by collinearity of only one of the principal directions of strain with any one of the material symmetry axes lying in a particular plane of elastic symmetry (plane Π, see Fig. 2). The stationarity points belonging to this class exist provided that the given principal strains fulfil certain inequalities involving the elastic constants. Finally, in the special case where the principal strains fulfil a linear constraint (depending on the elastic constants), stationarity can be achieved simply with one of the principal directions of strain lying in the plane of elastic symmetry Π, without any collinearity with material symmetry axes. The order of the strain energy density values at the stationarity points is shown to depend on three material parameters, κ, κ' and γ.

The case of transverse isotropy (Section 5) has been studied as a special sub-case of tetragonal symmetry. The presence of a plane of elastic isotropy Π (i.e., of an axis of rotational symmetry) simplifies the discussion. Only two classes of critical points exist: one in which the axis of rotational symmetry coincides with any one of the principal directions of strain; the other one, characterized by one of the principal directions of strain lying in the plane Π. For transverse isotropy, only one material parameter, κ, governs the order of the values of the energy density at the stationarity points.

For both classes of solids considered above, the plane of elastic symmetry Π is found to contain always at least one of the principal directions of strain.

Substantial differences emerge when the cubic symmetry is dealt with (Section 6). Beside the solutions corresponding to complete collinearity, stationarity of the strain energy density can also be achieved, *for any state of strain*, when none of the principal directions of strain (and stress) lies in any one of the elastic mirror symmetry planes. Only one material parameter, γ, rules the order of the values of the strain energy density corresponding to the two classes of solutions identified.

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Appendix. Singularity of $\boldsymbol{A}_{BA}$ for the cubic symmetry

The necessary condition for stationarity of the strain energy density is given by Eq. (32), and reads $\det \boldsymbol{A}_{BA} = 0$. $\boldsymbol{A}_{BA}$ is the lower left square unsymmetric part of:

$$\boldsymbol{A} = \boldsymbol{q}\widehat{\boldsymbol{A}}\boldsymbol{q}^{\mathrm{T}}. \tag{106}$$

In the case of cubic symmetry, this expression can be written in symbolic form as

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{q}_{AA} & \boldsymbol{q}_{AB} \\ \boldsymbol{q}_{BA} & \boldsymbol{q}_{BB} \end{pmatrix} \begin{pmatrix} \widehat{\boldsymbol{A}}_{AA} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{q}_{AA}^{\mathrm{T}} & \boldsymbol{q}_{BA}^{\mathrm{T}} \\ \boldsymbol{q}_{AB}^{\mathrm{T}} & \boldsymbol{q}_{BB}^{\mathrm{T}} \end{pmatrix}. \tag{107}$$

Thus, the necessary condition for stationarity is

$$\det \boldsymbol{A}_{BA} = \det(\boldsymbol{q}_{BA}\widehat{\boldsymbol{A}}_{AA}\boldsymbol{q}_{AA}^{\mathrm{T}}) = 0, \tag{108}$$

where

$$\widehat{\boldsymbol{A}}_{AA} = \mathrm{diag}(\hat{a}_{11}, \hat{a}_{11}, \hat{a}_{11}) \tag{109}$$

and matrices $\boldsymbol{q}_{BA}$ and $\boldsymbol{q}_{AA}$ are given by Eqs. (14) and (12) in terms of the Cartesian components Q_{ij} of the rotation tensor in three dimensions. By exploiting the matrix product in (108), the elements of the first row of matrix $\boldsymbol{A}_{BA}$ turn out to be:

$$a_{41} = \sqrt{2}\hat{a}_{11}(Q_{21}Q_{31}Q_{11}^2 + Q_{22}Q_{32}Q_{12}^2 + Q_{23}Q_{33}Q_{13}^2), \tag{110}$$

$$a_{42} = \sqrt{2}\hat{a}_{11}(Q_{21}^3Q_{31} + Q_{22}^3Q_{32} + Q_{23}^3Q_{33}), \tag{111}$$

$$a_{43} = \sqrt{2}\hat{a}_{11}(Q_{21}Q_{31}^3 + Q_{22}Q_{32}^3 + Q_{23}Q_{33}^3). \tag{112}$$

From the orthogonality condition $\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{Q} = \mathbf{1}$, the following expressions can be obtained:

$$Q_{31}^2 = 1 - Q_{11}^2 - Q_{21}^2, \tag{113}$$

$$Q_{32}^2 = 1 - Q_{12}^2 - Q_{22}^2, \tag{114}$$

$$Q_{33}^2 = 1 - Q_{13}^2 - Q_{23}^2, \tag{115}$$

which, substituted into Eq. (112), easily furnish

$$a_{43} = \sqrt{2}\hat{a}_{11}\{(Q_{21}Q_{31} + Q_{22}Q_{32} + Q_{23}Q_{33}) - (Q_{21}Q_{31}Q_{11}^2 + Q_{22}Q_{32}Q_{12}^2 + Q_{23}Q_{33}Q_{13}^2) - (Q_{21}^3Q_{31} + Q_{22}^3Q_{32} + Q_{23}^3Q_{33})\}. \tag{116}$$

By virtue of orthogonality $\boldsymbol{QQ}^{\mathrm{T}} = \mathbf{1}$, the first term in round brackets in (116) vanishes, whereas the second and third terms in brackets are equal to a_{41} and a_{42} respectively. Therefore,

$$a_{43} = -(a_{41} + a_{42}). \tag{117}$$

Repeating the same procedure for the second and third rows of matrix $\boldsymbol{A}_{BA}$, the following equalities are accordingly obtained:

$$a_{53} = -(a_{51} + a_{52}), \quad a_{63} = -(a_{61} + a_{62}). \tag{118}$$

Thus, in the cubic case, the third column of matrix $\boldsymbol{A}_{BA}$ is equal to minus the sum of the first two, and consequently this shows that $\det \boldsymbol{A}_{BA}$ is always equal to 0.

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